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Inverse Optimality for Extremum Seeking Feedback under Delays ⋆

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Abstract: We present a Gradient-based extremum seeking algorithm for maximizing unknown maps in the presence of constant delays. It is incorporated a filtered predictor feedback with a perturbation-based estimate for the Hessian of locally quadratic maps. Exponential stability and convergence to a small neighborhood of the unknown extremum point are achieved by using backstepping transformation and averaging theory in infinite dimensions. The low-pass filter (with a high enough pole) in the predictor feedback allows the technical application of the Hale and Lunel's averaging theorem for functional differential equations and also establishes an inverse optimal result for the closed-loop system. This inverse optimality property is for the first time demonstrated in extremum seeking designs and justifies the heuristic use of a low-pass filter between the demodulation and the integrator, which has historically been a part of the extremum seeking implementations free of delays.

Keywords: inverse optimality, time delay, adaptive control, extremum seeking, predictors, backstepping transformation, averaging in infinite dimensions.

1. INTRODUCTION

In extremum seeking (ES) (Krstić and Wang, 2000), there are many publications applying high-pass and low-pass filters in order to improve the closed-loop system performance and to facilitate the tuning parameters (Adetola and Guay, 2007; Tan et al., 2009; Nesić et al., 2010; Ghaffari et al., 2012; Liu and Krstić, 2012). However, they do not present any theoretical support that justifies the inclusion of such filters, on the contrary, only heuristic arguments are given.

In this paper, for the first time in the literature, the proof of inverse optimality and its influence on the extremum seeking feedback is discussed in the presence of delays (although the results are also valid in the case without delays). We show that the basic predictor feedback controller originally proposed in (Oliveira and Krstić, 2015; Oliveira et al., 2017), when applied through a low-pass filter, is inverse optimal and study its robustness to the low-pass filter time constant.

Inverse optimality was defined by (Kalman, 1964) as follows: "Given a dynamic system and a known control law, find performance criteria (if any) for which this control law is optimum". Inverse optimality can be related to a Lyapunov concept, with a special control law. In this sense, the inverse optimality is guaranteed when a stabilizing controller is optimal for some criteria and, for a given Lyapunov function, it is possible to show the feedback law is optimal with respect to some cost function. In general, this functional includes a control input penalty (or in its derivative or rate) and has an infinite gain margin (Krstić, 2008; Cai et al., 2018).

Notice that, so far ES was neither studied with Lyapunov tools nor has a control input whose cost should be optimized oven infinite time. In this paper, we study the inverse optimality of the average system, which we do via Lyapunov method, and we also want to minimize the update rate over the infinite interval. Results and simulations illustrate the advantages of satisfying the inverse optimality such as improved closed-loop responses.

Norms and Notations: The 2−norm of the state vector $X(t)$ for a finite-dimensional system described by an Ordinary Differential Equation (ODE) is denoted by single bars, $|X(t)|$. In contrast, norms of functions (of x) are denoted by double bars. By default, $\|\cdot\|$ denotes the spatial $L_2[0,D]$ norm, *i.e.*, $\|\cdot\| = \|\cdot\|_{L_2[0,D]}$. Since the state variable $u(x,t)$ of the infinite-dimensional system governed by a Partial Differential Equation (PDE) is a function of two arguments, we should emphasize that taking a norm in one of the variables makes the norm a function of the other variable. For example, the $L_2[0,D]$ norm of $u(x,t)$ in $x \in [0,D]$ is $\|u(t)\| = \left(\int_0^D u^2(x,t)dx\right)^{1/2}$, see (Krstić, 2009). The partial derivatives of $u(x,t)$ are denoted by $u_t(x,t)$ and $u_x(x,t)$ or, occasionally, by $\partial_t u_{\rm av}(x,t)$ and $\partial_x u_{\rm av}(x,t)$ to refer the operator for its average signal $u_{\rm av}(x,t)$. As

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defined in (Khalil, 2002), big $\mathcal{O}(\epsilon)$ notation is used to quantify approximations or order of magnitude relation of vector functions, valid for "ϵ sufficiently small".

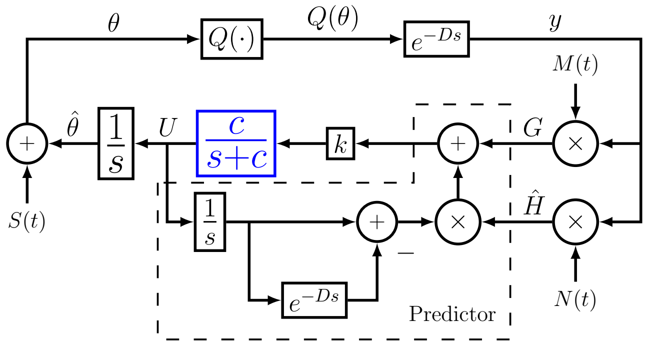

Fig. 1. Block diagram of the basic prediction scheme for output-delay compensation in gradient ES. The low-pass filter (in blue) is the key element employed to guarantee the inverse optimality.

2. EXTREMUM SEEKING UNDER DELAYS

The goal of scalar ES is to maximize (or minimize) the output $y \in R$ of an unknown nonlinear static map $Q(\theta)$ by changing its input $\theta \in R$. We further assume there is a *constant and known* delay $D \geq 0$ in the measurement system such that the delayed output signal is

$$y(t) = Q(\theta(t-D)) \,. \tag{1}$$

For the sake of clarity, we illustrate the complete output-delayed system in the block diagram of Figure 1. Without loss of generality, we also consider the maximum seeking problem. For the sake of simplicity, we assume the nonlinear map to be optimized is quadratic as

$$Q(\theta) = y^* + \frac{H}{2}(\theta - \theta^*)^2 \,, \tag{2}$$

with $\theta^* \in R$, $y^* \in R$ and $H \in R$ being unknown scalar constants. The maximizer of the input parameter θ is given by θ^*, while y^* is the extremum point and $H < 0$ is the unknown Hessian of the map.

Substituting (2) into (1), lead us to the following quadratic static map under delay:

$$y(t) = y^* + \frac{H}{2}(\theta(t-D) - \theta^*)^2 \,. \tag{3}$$

2.1 Signals and Systems

Let us define the the *estimation error*:

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta^* \tag{4}$$

where $\hat{\theta}$ is the estimate of θ^*. According to Figure 1, one can write the *error dynamics* as

$$\dot{\tilde{\theta}}(t-D) = U(t-D) \,. \tag{5}$$

In addition, one gets

$$G(t) = M(t)y(t) \,, \quad \theta(t) = \hat{\theta}(t) + S(t) \,, \tag{6}$$

with additive and multiplicative dithers given by

$$S(t) = a\sin(\omega(t+D)) \,, \quad M(t) = \frac{2}{a}\sin(\omega t) \,. \tag{7}$$

In (7), the amplitude a and frequency ω must be nonzero, assuming small and large values (as discussed later on), respectively.

The estimate of the unknown Hessian H is given by

$$\hat{H}(t) = N(t)y(t) \,, \tag{8}$$

with the demodulating signal $N(t)$ being

$$N(t) = -\frac{8}{a^2}\cos(2\omega t) \,. \tag{9}$$

From (Ghaffari et al., 2012), it is possible to show

$$\frac{1}{\Pi}\int_0^{\Pi} N(\sigma)y d\sigma = H \,, \quad \Pi = 2\pi/\omega \,, \tag{10}$$

for a quadratic map, as assumed in (2). Basically, the expression (9) give us an averaging-based estimate of H by means of (10), i.e., $\hat{H}_{\rm av} = (Ny)_{\rm av} = H$.

2.2 Predictor Feedback with Averaging-based Estimates

From the averaging analysis, we can obtain the average version of $G(t)$ in (6) as follows

$$G_{\rm av}(t) = H\tilde{\theta}_{\rm av}(t-D) \,. \tag{11}$$

Moreover, from (5), we can also derive the average models

$$\dot{\tilde{\theta}}_{\rm av}(t-D) = U_{\rm av}(t-D) \,, \tag{12}$$

$$G_{\rm av}(t) = HU_{\rm av}(t-D) \,, \tag{13}$$

with $U_{\rm av} \in R$ being the average version of the control signal $U \in R$.

As shown in (Oliveira and Krstić, 2015; Oliveira et al., 2017), by computing the variation of constants formula of (13), we can write the future state as

$$G_{\rm av}(t+D) = G_{\rm av}(t) + H\int_{t-D}^{t} U_{\rm av}(\sigma)d\sigma \,, \tag{14}$$

which is given in terms of the average control signal $U_{\rm av}(\sigma)$ from the past window $[t-D, t]$. It results in the next predictor-feedback law

$$U_{\rm av}(t) = k\left[G_{\rm av}(t) + H\int_{t-D}^{t} U_{\rm av}(\sigma)d\sigma\right] \,, \quad k > 0 \,, \tag{15}$$

which is able to make the equilibrium $\tilde{\theta}^e_{\rm av} = 0$ of the closed-loop system (12) and (15) exponentially stable.

As in In (Oliveira and Krstić, 2015; Oliveira et al., 2017), the following infinite-dimensional filtered predictor feedback, computed from its non average version (15), is again considered:

$$U(t) = \frac{c}{s+c}\left\{k\left[G(t) + \hat{H}(t)\int_{t-D}^{t} U(\tau)d\tau\right]\right\} \tag{16}$$

where $c > 0$ is a design constant chosen sufficiently large.

This low-pass filtering was particularly required in the stability analysis of our earlier publications (Oliveira and Krstić, 2015; Oliveira et al., 2017) when the averaging theorem in infinite dimensions was invoked (Hale and Lunel, 1990).

Now, in the next section, we are going to demonstrate the advantages of such a filtering procedure go beyond to merely solve technical limitations in the analysis, but it can also improved the control performance of the closed-loop ES system, which is rigorously justified through the concept of inverse optimality (Kalman, 1964).

3. INVERSE OPTIMAL DESIGN

In this section, the stability analysis is carried out and the proof of inverse optimality is presented.

Theorem 1. There exists c^* such that the average feedback system of (5) and (16) is exponentially stable in the sense of the norm

$$\Psi(t) = \left(|\tilde{\theta}_{\text{av}}(t-D)|^2 + \int_{t-D}^{t} U_{\text{av}}(\tau)^2 d\tau + U_{\text{av}}(t)^2\right)^{1/2}$$

for all $c > c^*$. Furthermore, there exists $c^{**} > c^*$ such that for any $c \geq c^{**}$, the feedback (16) minimizes the cost functional

$$J = \int_0^\infty (\mathcal{L}(t) + \dot{U}_{\text{av}}^2(t))dt, \quad (17)$$

where $\mathcal{L}(t)$ is a functional of $(\tilde{\theta}_{\text{av}}(t-D),\ U(\tau))$, $\tau \in [t-D, t]$ and such that

$$\mathcal{L}(t) \geq \mu\Psi(t)^2 \quad (18)$$

for some $\mu(c) > 0$ with a property that $\mu(c) \to \infty$ as $c \to \infty$.

Proof. The proof is structured into ***Step 1*** to ***Step 6***.

Step 1: *Transport PDE for Delay Representation*

Considering (Krstić, 2009), the delay in (5) is represented using a transport PDE such as

$$\dot{\tilde{\theta}}(t-D) = u(0,t)\,, \quad (19)$$

$$u_t(x,t) = u_x(x,t)\,, \quad x \in [0,D]\,, \quad (20)$$

$$u(D,t) = U(t)\,, \quad (21)$$

with the solution of (20)–(21) being

$$u(x,t) = U(t+x-D)\,. \quad (22)$$

Step 2: *Average Model of the Closed-loop System*

By denoting

$$\tilde{\vartheta}(t) := \tilde{\theta}(t-D)\,, \qquad \tilde{\vartheta}_{\text{av}}(t) = \tilde{\theta}_{\text{av}}(t-D)\,, \quad (23)$$

the average version of system (19)–(21), with $U(t)$ in (16) is given by:

$$\dot{\tilde{\vartheta}}_{\text{av}}(t) = u_{\text{av}}(0,t)\,, \quad (24)$$

$$\partial_t u_{\text{av}}(x,t) = \partial_x u_{\text{av}}(x,t)\,, \quad x \in [0,D]\,, \quad (25)$$

$$\frac{d}{dt}u_{\text{av}}(D,t) = -cu_{\text{av}}(D,t) + ckH\left[\tilde{\vartheta}_{\text{av}}(t) + \int_0^D u_{\text{av}}(\sigma,t)d\sigma\right], \quad (26)$$

where the filter $c/(s+c)$ in (16) was also represented in the state-space form. The solution of the transport PDE (25)–(26) is given by

$$u_{\text{av}}(x,t) = U_{\text{av}}(t+x-D)\,. \quad (27)$$

Step 3: *Backstepping Transformation, its Inverse and the Target System*

Since we are not able to prove directly the stability for the average closed-loop system (24)–(26), we consider the infinite-dimensional backstepping transformation of the delay state

$$w(x,t) = u_{\text{av}}(x,t) - kH\left[\tilde{\vartheta}_{\text{av}}(t) + \int_0^x u_{\text{av}}(\sigma,t)d\sigma\right], \quad (28)$$

with inverse given by

$$u_{\text{av}}(x,t) = w(x,t) + kH\left[e^{kHx}\tilde{\vartheta}_{\text{av}}(t) + \int_0^x e^{kH(x-\sigma)}w(\sigma,t)d\sigma\right]. \quad (29)$$

The transformation (28) maps the system (24)–(26) into the target system:

$$\dot{\tilde{\vartheta}}_{\text{av}}(t) = kH\tilde{\vartheta}_{\text{av}}(t) + w(0,t)\,, \quad (30)$$

$$w_t(x,t) = w_x(x,t)\,, \quad x \in [0,D]\,, \quad (31)$$

$$w(D,t) = -\frac{1}{c}\partial_t u_{\text{av}}(D,t)\,. \quad (32)$$

Step 4: *Lyapunov-Krasovskii Functional*

Consider the following Lyapunov functional

$$V(t) = \frac{\tilde{\vartheta}_{\text{av}}^2(t)}{2} + \frac{a}{2}\int_0^D (1+x)w^2(x,t)dx + \frac{1}{2}w^2(D,t)\,, \quad (33)$$

where the parameter $a = -\frac{1}{KH}$ and $kH < 0$. Computing the time-derivative of (33) along with (30)–(32), we have

$$\begin{aligned}\dot{V}(t) &= kH\tilde{\vartheta}_{\text{av}}^2(t) + \tilde{\vartheta}_{\text{av}}(t)w(0,t)\\ &\quad + a\int_0^D (1+x)w(x,t)w_x(x,t)dx + w(D,t)w_t(D,t)\\ &= kH\tilde{\vartheta}_{\text{av}}^2(t) + \tilde{\vartheta}_{\text{av}}(t)w(0,t) + \frac{a(1+D)}{2}w^2(D,t)\\ &\quad - \frac{a}{2}w^2(0,t) - \frac{a}{2}\int_0^D w^2(x,t)dx + w(D,t)w_t(D,t) \quad (34)\\ &\leq kH\tilde{\vartheta}_{\text{av}}^2(t) + \frac{\tilde{\vartheta}_{\text{av}}^2(t)}{2a} - \frac{a}{2}\int_0^D w^2(x,t)dx\\ &\quad + w(D,t)\left[w_t(D,t) + \frac{a(1+D)}{2}w(D,t)\right].\end{aligned}$$

Now, following the same procedure given in (Oliveira and Krstić, 2015), we get

$$\dot{V}(t) \leq -\frac{1}{4a}\tilde{\vartheta}_{\rm av}^2(t) - \frac{a}{4(1+D)}\int_0^D (1+x)w^2(x,t)dx - (c-c^*)w^2(D,t)\,, \tag{35}$$

where

$$c^* = \frac{a(1+D)}{2} - kH + a\left|(kH)^2 e^{kHD}\right|^2 + \frac{1}{a}\left\|(kH)^2 e^{kH(D-\sigma)}\right\|^2 . \tag{36}$$

According to (36), an upper bound for c^* can be obtained from the known delay D as well as some lower and upper bounds of the Hessian H. Thus, from (35), if we chose c such that $c > c^*$, we arrive at

$$\dot{V}(t) \leq -\mu^* V(t)\,, \tag{37}$$

for some $\mu^* > 0$. Hence, the closed-loop system is exponentially stable in the sense of the full-state norm

$$\left(|\tilde{\vartheta}_{\rm av}(t)|^2 + \int_0^D w^2(x,t)dx + w^2(D,t)\right)^{1/2}, \tag{38}$$

i.e., in the transformed variable $(\tilde{\vartheta}_{\rm av}\,, w)$.

Step 5: *Average System Exponential Stability Estimate (in L_2 norm)*

In order to assure exponential stability for the average system (24)–(26) in the sense of the norm

$$\left(|\tilde{\vartheta}_{\rm av}(t)|^2 + \int_0^D u_{\rm av}^2(x,t)dx + u_{\rm av}^2(D,t)\right)^{1/2},$$

we need to show there exist constants $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$\alpha_1\Psi(t) \leq V(t) \leq \alpha_2\Psi(t)\,, \tag{39}$$

where $\Psi(t):=|\tilde{\vartheta}_{\rm av}(t)|^2+\int_0^D u_{\rm av}^2(x,t)dx+u_{\rm av}^2(D,t)$, or using (23) and (27),

$$\Psi(t) := |\tilde{\theta}_{\rm av}(t-D)|^2 + \int_{t-D}^t U_{\rm av}^2(\tau)d\tau + U_{\rm av}^2(t)\,. \tag{40}$$

The inequality (39) can be directly established from (28), (29), (33), by using the Cauchy-Schwartz inequality and other calculations, such as in the proof of Theorem 2.1 in (Krstić, 2009). Thus, taking into account (37), we obtain

$$\Psi(t) \leq \frac{\alpha_2}{\alpha_1}e^{-\mu t}\Psi(0)\,, \tag{41}$$

which concludes the proof of exponential stability in the original variables $(\tilde{\vartheta}_{\rm av}\,, u_{\rm av})$.

Step 6: *Inverse Optimality*

Based on the proof of Theorem 6 in (Smyshlyaev and Krstic, 2004) and Theorem 2.8 in (Krstic and Deng, 1999), we chose $c^{**} = 4c^*$, $c = 2c^*$ and define $\mathcal{L}(t)$ as:

$$\mathcal{L}(t) = -2c\dot{V}(t) + c(c-4c^*)w^2(D,t) \geq c\left(\frac{1}{2}k\tilde{\vartheta}_{\rm av}^2(t) + \frac{a}{2}\int_0^D w^2(x,t)dx + (c-2c^*)w^2(D,t)\right) \tag{42}$$

where $\vartheta_{\rm av}(t) := \tilde{\theta}_{\rm av}(t-D)$, according to (23).

Using (28) for $x = D$ and the fact that $u_{\rm av}(D,t) = U_{\rm av}(t)$, from (32) we get (26). Let us now consider $w(D,t)$. From (28) and (29), it is easy to see that

$$w_t(D,t) = \partial_t u_{\rm av}(D,t) - kHu_{\rm av}(D,t)\,, \tag{43}$$

where $\partial_t u_{\rm av}(D,t) = \dot{U}_{\rm av}(t)$. Plugging (32) and (29) into (43), we get

$$w_t(D,t) = -cw(D,t) - kHw(D,t) -(kH)^2\left[e^{kHD}\tilde{\vartheta}_{\rm av}(t) + \int_0^D e^{kH(D-\sigma)}w(\sigma,t)d\sigma\right]. \tag{44}$$

By plugging (44) into derivative of the Lyapunov functional (34), one has

$$\begin{aligned}\dot{V}(t) &= kH\tilde{\vartheta}_{\rm av}^2(t) + \tilde{\vartheta}_{\rm av}(t)w(0,t) + \frac{a(1+D)}{2}w^2(D,t) \\ &-\frac{a}{2}w^2(0,t) - \frac{a}{2}\int_0^D w^2(x,t)dx - 2c^*w^2(D,t) \\ &-kHw^2(D,t) - (kH)^2 w(D,t)e^{kHD}\tilde{\vartheta}_{\rm av}(t) \\ &-(kH)^2 w(D,t)\int_0^D e^{kH(D-\sigma)}w(\sigma,t)d\sigma\,.\end{aligned} \tag{45}$$

Then, by applying (45) to (42), $\mathcal{L}(t)$ can be written as:

$$\begin{aligned}\mathcal{L}(t) &= -2ckH\tilde{\vartheta}_{\rm av}^2(t) - 2c\tilde{\vartheta}_{\rm av}(t)w(0,t) - 2c\frac{a(1+D)}{2}w^2(D,t) \\ &+caw^2(0,t) + ca\int_0^D w^2(x,t)dx + 2ckHw^2(D,t) \\ &+2c(kH)^2 w(D,t)e^{kHD}\tilde{\vartheta}_{\rm av}(t) \\ &+2c(kH)^2 w(D,t)\int_0^D e^{kH(D-\sigma)}w(\sigma,t)d\sigma + c^2w^2(D,t).\end{aligned} \tag{46}$$

On the other hand, substituting the average version of the system (24) into the target system (30), we obtain

$$u_{\rm av}(0,t) = kH\tilde{\vartheta}_{\rm av}(t) + w(0,t). \tag{47}$$

Rearranging (47) in order to isolate $w(0,t)$, we can write:

$$w(0,t) = u_{\rm av}(0,t) - kH\tilde{\vartheta}_{\rm av}(t). \tag{48}$$

Then, plugging (48) into (46), and adding-subtracting the term $\gamma\tilde{\vartheta}_{\rm av}^2(t)$ (in blue) in the right-hand side of the resulting equation, lead us to

$$\mathcal{L}(t) = c\Big(a(kH)^2\tilde{\vartheta}_{\rm av}^2(t) - 2(akH+1)u_{\rm av}(0,t)\tilde{\vartheta}_{\rm av}(t)$$

$$-a(1+D)w^2(D,t)-\gamma\tilde{\vartheta}^2_{\rm av}(t)+2(kH)^2w(D,t)$$
$$\times\Big[e^{kHD}\tilde{\vartheta}_{\rm av}(t)+\int_0^D e^{kH(D-\sigma)}w(\sigma,t)d\sigma\Big]\Big)$$
$$+au^2_{\rm av}(0,t)+\frac{a}{2}\int_0^D w^2(x,t)dx+w^2(D,t)(2c^*+2kH)$$
$$+c\Big(\gamma\tilde{\vartheta}^2_{\rm av}(t)+\frac{a}{2}\int_0^D w^2(x,t)dx+(c-2c^*)w^2(D,t)\Big). \quad (49)$$

Reminding that $a=-\frac{1}{kH}$, and replacing kH by $-\frac{1}{a}$ in (49), one has

$$\mathcal{L}(t)=c\Big(\Big[\frac{1}{a}-\gamma\Big]\tilde{\vartheta}^2_{\rm av}(t)\big(2c^*-a(1+D)-\frac{2}{a}\big)w^2(D,t)$$
$$+au^2_{\rm av}(0,t)+\frac{a}{2}\int_0^D w^2(x,t)dx+\frac{2}{a^2}w(D,t)$$
$$\times\Big[e^{kHD}\tilde{\vartheta}_{\rm av}(t)+\int_0^D e^{kH(D-\sigma)}w(\sigma,t)d\sigma\Big]\Big)$$
$$+c\Big(\gamma\tilde{\vartheta}^2_{\rm av}(t)+\frac{a}{2}\int_0^D w^2(x,t)dx+(c-2c^*)w^2(D,t)\Big). \quad (50)$$

After some mathematical manipulations, the term $\mathcal{L}(t)$ in (50) can be rewritten as:

$$\mathcal{L}(t)=\Upsilon(D,t)+c\big(\gamma\tilde{\vartheta}^2_{\rm av}(t)+\frac{a}{2}\int_0^D w^2(x,t)dx$$
$$+(c-2c^*)w^2(D,t)\big)\,, \quad (51)$$

where $\Upsilon(D,t)$ is given by:

$$\Upsilon(D,t)=c\Big(\Big[\frac{1}{a}-\gamma\Big]\tilde{\vartheta}^2_{\rm av}(t)+\big(2c^*-a(1+D)-\frac{2}{a}\big)w^2(D,t)$$
$$+au^2_{\rm av}(0,t)+\frac{a}{2}\int_0^D w^2(\sigma,t)d\sigma+\frac{2}{a^2}w(D,t)e^{kHD}\tilde{\vartheta}_{\rm av}(t)$$
$$+\frac{2}{a^2}w(D,t)\int_0^D e^{kH(D-\sigma)}w(\sigma,t)d\sigma\Big)\,. \quad (52)$$

In order to satisfy inequality (42), it is necessary to ensure $\Upsilon(D,t)\geq 0$. To assure the latter condition, we will analyze the terms in (52) with undefined signs so that we can guarantee they are non negative. After adding and subtracting the terms $\frac{1}{a^2}[\tilde{\vartheta}^2_{\rm av}+w^2(D,t)]$ and $\frac{2\sqrt{D}}{a^2}[w^2(D,t)+\int_0^D w^2(\sigma,t)d\sigma]$ (in blue and red) into (52), $\Upsilon(D,t)$ can be rewritten as:

$$\Upsilon(D,t)=c\Big(\Big[\frac{1}{a}-\frac{1}{a^2}-\gamma\Big]\tilde{\vartheta}^2_{\rm av}(t)$$
$$+\big(2c^*-a(1+D)-\frac{2}{a}-\frac{1}{a^2}-\frac{2\sqrt{D}}{a^2}\big)w^2(D,t)$$
$$+au^2_{\rm av}(0,t)+\Big[\frac{a}{2}-\frac{2\sqrt{D}}{a^2}\Big]\int_0^D w^2(\sigma,t)d\sigma$$
$$+\frac{2}{a^2}w(D,t)e^{kHD}\tilde{\vartheta}_{\rm av}(t)+\frac{1}{a^2}w^2(D,t)+\frac{1}{a^2}\tilde{\vartheta}^2_{\rm av}(t)$$
$$+\frac{2}{a^2}w(D,t)\int_0^D e^{kH(D-\sigma)}w(\sigma,t)d\sigma$$
$$+\frac{2\sqrt{D}}{a^2}w^2(D,t)+\frac{2\sqrt{D}}{a^2}\int_0^D w^2(\sigma,t)d\sigma\Big). \quad (53)$$

By employing the Young and Cauchy-Schwartz inequalities, it is possible verify valid lower bounds for the terms which were added and subtracted in (53), so that:

$$\frac{1}{a^2}w^2(D,t)+\frac{1}{a^2}\tilde{\vartheta}^2_{\rm av}(t)\geq\frac{2}{a^2}\Big|w(D,t)e^{kHD}\tilde{\vartheta}_{\rm av}(t)\Big|, \quad (54)$$

$$\frac{2\sqrt{D}}{a^2}\Big(w^2(D,t)+\int_0^D w^2(\sigma,t)d\sigma\Big)$$
$$\geq\frac{2}{a^2}\Big|w(D,t)\int_0^D e^{kH(D-\sigma)}w(\sigma,t)d\sigma\Big|. \quad (55)$$

Analyzing $\Upsilon(D,t)$ in terms of the lower bounds in (54) and (55), we get

$$\Upsilon(D,t)\geq c\Big(\Big[\frac{1}{a}-\frac{1}{a^2}-\gamma\Big]\tilde{\vartheta}^2_{\rm av}(t)$$
$$+\big(2c^*-a(1+D)-\frac{2}{a}-\frac{1}{a^2}-\frac{2\sqrt{D}}{a^2}\big)w^2(D,t)$$
$$+au^2_{\rm av}(0,t)+\Big[\frac{a}{2}-\frac{2\sqrt{D}}{a^2}\Big]\int_0^D w^2(\sigma,t)d\sigma$$
$$+\frac{2}{a^2}w(D,t)e^{kHD}\tilde{\vartheta}_{\rm av}(t)+\frac{2}{a^2}\Big|w(D,t)e^{kHD}\tilde{\vartheta}_{\rm av}(t)\Big|$$
$$+\frac{2}{a^2}w(D,t)\int_0^D e^{kH(D-\sigma)}w(\sigma,t)d\sigma$$
$$+\frac{2}{a^2}\Big|w(D,t)\int_0^D e^{kH(D-\sigma)}w(\sigma,t)d\sigma\Big|\Big). \quad (56)$$

Then, to ensure $\Upsilon(D,t)\geq 0$ it is necessary to satisfy the following conditions:

<u>$1^{\rm st}$ **Condition:**</u>

$$\frac{1}{a}-\frac{1}{a^2}-\gamma>0\,,\qquad \gamma<\frac{a-1}{a^2}$$

<u>$2^{\rm nd}$ **Condition:**</u>

Reminding that $c=2c^*$,

$$2c^*-a(1+D)-\frac{2}{a}-\frac{1}{a^2}-\frac{2\sqrt{D}}{a^2}>0$$
$$c>a(1+D)+\frac{2}{a}+\frac{1}{a^2}+\frac{2\sqrt{D}}{a^2}$$

3rd Condition:

$$\frac{a}{2} - \frac{2\sqrt{D}}{a^2} > 0\,, \qquad a > \sqrt[3]{4\sqrt{D}}$$

Therefore, considering $\mathcal{L}(t)$ given in (51) and $\Upsilon(D,t)$ given in (52), under the conditions imposed for γ, a and c, one can conclude $\Upsilon(D,t) \geq 0$ and

$$\mathcal{L}(t) \geq c\left(\frac{1}{2}k\tilde{\vartheta}^2_{\rm av}(t) + \frac{a}{2}\int_0^D w^2(x,t)dx + (c-2c^*)w^2(D,t)\right),$$

with $\gamma = k/2$.

Hence, we have $\mathcal{L}(t) \geq \mu\Psi(t)^2$, for the same reason that (39) holds, completing the proof of inverse optimality. □

Finally, analogously to the Steps 6 and 7 performed for the proof of Theorem 1 in (Oliveira and Krstić, 2015), we can invoke the averaging theorem in infinite dimensions by Hale and Lunel (1990) and still conclude the results for constant delays, where the estimation errors $\theta(t) - \theta^*$ and $y(t) - y^*$ are ultimately of order $\mathcal{O}(a + 1/\omega)$ and $\mathcal{O}(a^2 + 1/\omega^2)$, respectively.

4. NUMERICAL SIMULATIONS

In order to evaluate the effects of the inverse optimality for the ES feedback under delays, the next quadratic map (1)–(2) is considered: $Q(\theta) = 5 - 0.1(\theta - 3)^2$, with an output delay of $D = 5\,$s. The extremum point is $(\theta^*;\ y^*) = (3;\ 5)$ and the Hessian of the corresponding static map is $H = -0.1$. For the simulation tests, the following parameters were employed: $\omega = 10$ rad/s , $k = 0.8$, $\hat{\theta}(0) = -5$ and $a = 0.2$. The time constant of the low-pass filter $c = 40$ was chosen to satisfy the Conditions 1-3 in the ***Step 6*** of the proof of Theorem 1.

Figure 2 presents a numerical comparison between the ES fundamental variables with and without using the filter $\frac{c}{s+c}$ in feedback law (16). As it can be observed, in the first case where the inverse optimality is guaranteed, the input signal $\theta(t)$ converges monotonically rather than swinging up-and-down, thus improving the transient responses.

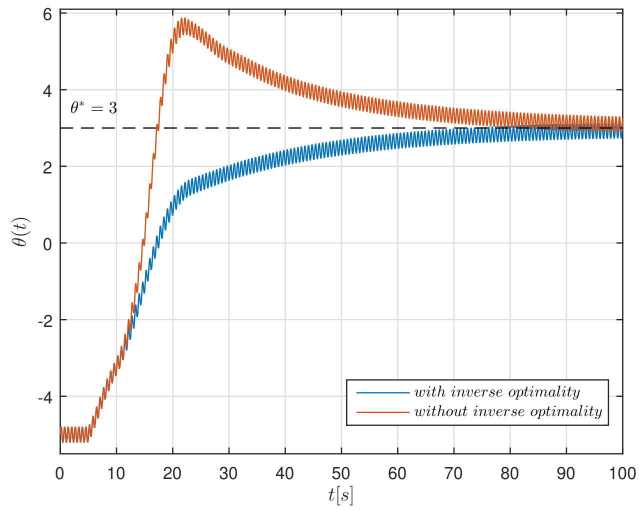

Fig. 2. ES with and without inverse optimality: input signal $\theta(t)$.

5. CONCLUSIONS

In this paper, we derived inverse optimality results for extremum seeking feedback with the low-pass filtered modification of the predictor-based feedback for delay compensation proposed in (Oliveira and Krstić, 2015). Extremum seeking is studied with Laypunov tools and has a control input whose cost can be optimized over infinite time. We have established the stability robustness to varying the parameter c from some large value c^* to ∞, recovering in the limit, the basic, unfiltered predictor-based feedback (15). The inverse optimality properties of the basic predictor feedback controller are illustrated by a numerical example.

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