

# Mean-field type quantum filter for a quantum Ising type system <sup>\*</sup>

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**Abstract:** Mean-field games or mean-field type control problems are one of distributed control schemes that reduce the computational burden. In this paper, a quantum version of mean-field game settings is developed and the mean-field type quantum filter is derived for quantum Ising models.

*Keywords:* mean-field games, quantum systems, mean-field type quantum filter, stochastic systems

## 1. INTRODUCTION

Mean-field games or mean-field type control problems are a kind of non-cooperative games introduced by Lasry and Lions (2007) and Huang et al. (2006, 2007), independently. Consider that  $N$  agents interact weakly with others. Each agent has a common cost function and they would like to optimize their cost without other agents' information. The situation is a typical non-cooperative game, so if each agent optimizes their cost, the optimal solution may be the Nash equilibrium solution under suitable assumptions. However, it is practically impossible to obtain the optimum if  $N$  is quite huge. Mean-field games can reduce this computational burden and give an approximate Nash equilibrium (Bensoussan et al. (2013); Carmona and Delarue (2018)).

In order to illustrate a mean-field type control problem, let us show an example as follows. There are  $N$  agents interacted through an empirical mean of them each other. Each agent has the following dynamics.

$$dx_i(t) = f(x_i(t), u_i(t), \bar{x}_N(t))dt + \sigma(x_i(t), u_i(t), \bar{x}_N(t))dw_i(t), \quad i = 1, \dots, N,$$

where  $\bar{x}_N(t) := \frac{1}{N} \sum_{j=1}^N x_j(t)$  and the functions  $f$  and  $g$  have certain suitable properties (e.g., Lipschitz continuous, boundedness, etc).  $\{w_i(t)\}$  is a standard Wiener process and for  $i \neq j$ ,  $w_i(t)$  and  $w_j(t)$  are independent. Moreover, all  $x_i(0)$  are independent and identically distributed. Assume that each agent  $i$  can only know  $x_i(t)$  up to  $t$ , so the agent  $i$  has  $\sigma$ -field  $\mathcal{F}_i(t) := \sigma\{x_i(\tau), \tau \in [0, t]\}$  at time  $t$ . The agent  $i$  would like to minimize a following cost under the other control laws  $u_{-i} := \{u_j(t) \mid t \in [0, t], j \neq i\}$ ;

$$J(x_i(0), u_i; u_{-i}) := \mathbb{E} \left[ \int_0^T C(x_i(t), u_i(t), \bar{x}_N(t))dt + \Phi(x_i(T)) \right].$$

In order to obtain an optimal feedback control, applying the Bellman's principle is a standard way and

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then the agents meet the  $N$  coupled Hamilton–Jacobi–Bellman(HJB) equations. Clearly, the optimization problem becomes very hard to solve if  $N$  is large.

On the other hand, if each  $x_i(0)$  are mutually independent, then we can expect that  $\bar{x}_N(t) \simeq \mathbb{E}_{x_j}[x_j(t) | \mathcal{F}_i(t)] = \mathbb{E}_{x_j}[x_j(t)]$  as  $N \rightarrow \infty$ , for  $j \neq i$ , where the expectation is taken under the suitable probability law. Hence, if  $N$  is enough large, it is a good idea to apply the law of large numbers for the empirical mean. Note that the assumption of independence is not assured for the finite number of agents because the systems have interaction through  $\bar{x}_N(t)$ . Fortunately, as  $N \rightarrow \infty$ ,  $x_i$  and  $x_j$  are asymptotically independent, so the law of large numbers holds in the asymptotical sense. This implies that if  $N$  is large enough, it is a good approximation to replace the empirical mean  $\bar{x}_N(t)$  by  $\mathbb{E}_{x_j}[x_j(t)]$  for any  $j \neq i$ . In order to calculate  $\mathbb{E}_{x_j}[x_j(t)]$  at each time  $t$ , we need to track the time evolution of their distribution called the Fokker–Planck(FP) equation. Consequently, the problem should be tackled with for agent  $i$  is *one* HJB equation coupled with *one* FP equation that describes the time development of the expectation. Therefore, each agent solves only the mean-field approximated optimal control problem. Since the HJB equation is a backward equation and the FP equation is a forward equation, there is time-inconsistency to solve the problem, which causes another difficulty to solve. However, the approximated optimal control problem requires much fewer equations to be solved. This situation occurs in many applications (Djehiche et al. (2017); Bauso (2017)), so the mean-field games, or mean-field type control problems, provide a good distributed-control scheme for some class of large dynamical systems. Of course, the approximated problem does not give an original optimal solution in general, however, approximation errors are analyzed by some research (See, e.g., Carmona and Delarue (2018); Cardaliaguet et al. (2019) and the references therein). If the state variables are not fully observed, then filtering techniques are necessary in addition to solving the above HJB and FP equations(Caines and Kizilkale (2017); Şen and Caines (2019)).

In this paper, we extend the notion of the mean-field type control problems to a certain class of open quantum systems and derive the mean-field type quantum filter. A motivational example of our purpose is quantum error correction, where the quantum system composes of qubits and is described by Ising model. Quantum error correction requires measurement-based feedback control for each qubit or logical qubit. A basic framework of measurement-based quantum feedback control is to estimate the quantum state via a quantum filter and then feed the input back to the system based on the estimate. However, since the total dimension of  $N$  spin systems becomes  $2^N$ , it is difficult to obtain the conditional quantum state or synthesis controller based on the model if  $N$  is large due to its computational burden. Therefore, some approximation is necessary to deal with such a large dimensional system. A low dimensional approximation of the quantum filter is considered by, e.g., van Handel and Mabuchi (2005); Nielsen et-al. (2009); Gao et al. (2019), and they work well for certain quantum systems. However, in some situations, the mean-field approximation can be natural. Quantum spin systems described by the Ising models are a good example to be analyzed by mean-field approximation (Schulz (1996); Sachdev and Bhatt (1990); Son and Vedral (2018)). However, as far as the author's knowledge, there is no quantum mean-field game or quantum mean-field type control problem. As the first step, we derive a quantum mean-field approximation for a quantum Ising model and mean-field type quantum filter.

Since the quantum filter is derived by Belavkin (1992), quantum filtering is important tool for quantum metrology and control (Wiseman and Milburn (2009)). There are several developments of quantum filtering, for example, quantum systems driven by single photon noise (Gough et al. (2013)), however, it has the same difficulty to implement the quantum filter in practice if the dimension of the system is very large. The proposed method in this paper will be useful for these quantum filters if the quantum systems have a mean-field type interaction.

The rest of this paper is organized as follows. In Section 2, the quantum probability theory, open quantum systems, and its quantum filter is introduced. In Section 3, a mean-field approximation of the quantum system is employed and derived the mean-field type quantum filter and this part is our main contribution. Conclusion and future work are in Section 4.

### Notations

$\mathbb{R}$  and  $\mathbb{C}$  are real numbers and complex numbers, respectively, and  $i := \sqrt{-1}$ .  $\mathcal{H}$  is a complex Hilbert space and we also denote  $\mathcal{H}_X$  if it is the Hilbert space of the system  $X$ . Any linear operator on a Hilbert space  $\mathcal{H}$  is denoted by hat, e.g.,  $\hat{X}$ . When positive operators  $\hat{X}$  and  $\hat{Y}$  satisfy  $\hat{X} = \hat{Y}^2$ , we denote  $\hat{Y} = \sqrt{\hat{X}}$ . The absolute value of operator is defined  $|\hat{X}| := \sqrt{\hat{X}^* \hat{X}}$ .  $\mathcal{L}(\mathcal{H})$  is a set of linear bounded operators on the Hilbert space  $\mathcal{H}$ .  $\hat{X} \geq 0$  means that  $\hat{X} \in \mathcal{L}(\mathcal{H})$  is a positive operator and  $\hat{X}^*$  implies the conjugate operator of  $\hat{X}$ .  $\text{Tr}[\bullet] : \mathcal{L}(\mathcal{H}) \rightarrow \mathbb{C}$  is the trace on linear operators.  $\mathcal{S}(\mathcal{H}) := \{\hat{\rho} \in \mathcal{L}(\mathcal{H}) \mid \hat{\rho} \geq 0, \text{Tr}[\hat{\rho}] = 1\}$  is a set of density operators.  $\hat{1}_{\mathcal{H}}$  is the identity opera-

tor on  $\mathcal{H}$  and we sometimes omit its subscript. Denote  $[\hat{X}, \hat{Y}]_{\pm} := \hat{X}\hat{Y} \pm \hat{Y}\hat{X}$ ,  $\forall \hat{X}, \hat{Y} \in \mathcal{L}(\mathcal{H})$ .  $\otimes$  represents the Kronecker product for matrices and the tensor product for operators or spaces.

## 2. BASICS OF OPEN QUANTUM SYSTEMS THEORY

Quantum systems are described by quantum probability theory. Here we briefly review the quantum theory. For details, see, e.g., Holevo (1982); Wiseman and Milburn (2009).

### 2.1 Quantum probability theory

Every quantum system is described by a suitably defined Hilbert space  $\mathcal{H}$ . All of the quantum physical quantities are denoted by self-adjoint operators on  $\mathcal{H}$ . In this paper, we only consider linear bounded operators except for quantum noise operators. We denote a set of linear bounded operators on  $\mathcal{H}$  by  $\mathcal{L}(\mathcal{H})$ . The observation of any quantum physical quantity is a randomly chosen number from the spectrum of the corresponding self-adjoint operator. Random outcomes of all bounded operators make the quantum statistics and the quantum expectation  $\mathbb{P}_{\hat{\rho}}$  is defined as  $\mathbb{P}_{\hat{\rho}}[\hat{X}] := \text{Tr}[\hat{\rho}\hat{X}]$ ,  $\hat{\rho} \in \mathcal{S}(\mathcal{H})$ . The quantum version of  $\sigma$ -measurable functions is von Neumann algebra, which is, roughly speaking, an algebra generated by projection operators with algebraic operations (Bouten et al. (2009)). Let  $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$  be a von Neumann subalgebra. A pair  $(\mathcal{A}, \mathbb{P}_{\hat{\rho}})$  is called *the quantum probability space*.

Different quantum systems have different Hilbert spaces, so if we would like to deal with them together, a tensor product Hilbert space is used. If there are  $\mathcal{H}_i$ ,  $i = 1, 2, \dots, N$ , then the whole system is described by

$$\mathcal{H}_{\text{tot}} := \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_N.$$

In particular, quantum fields are described by Fock spaces, which are a kind of uncountable tensor product Hilbert space. Then the linear operators on  $\mathcal{H}_i$  are as follows.

$$\hat{X}_i := \overbrace{\hat{1} \otimes \dots \otimes \hat{X} \otimes \dots \otimes \hat{1}}^{N\text{-fold tensor product}}_{i\text{-th}}$$

We also write  $\hat{X}_i \in \mathcal{L}(\mathcal{H}_i)$  for convenience. Note that  $[\hat{X}_i, \hat{Y}_j]_- = 0$  for all  $i \neq j$ .

There are several definitions of independence for quantum variables; see, e.g., Hora and Obata (2007) and the reference therein. In this paper, we use the commutative independence.

*Definition 1.* (Commutative independence).

Let  $\mathcal{H}$  is a Hilbert space composed of  $\mathcal{H}_i$ ,  $i = 1, \dots, m$  and  $(\mathcal{L}(\mathcal{H}), \mathbb{P}_{\hat{\rho}})$  be a quantum probability space. Systems  $i = 1, \dots, m$  are called *mutually commutative independent* if

$$\mathbb{P}_{\hat{\rho}}[\hat{X}_1 \dots \hat{Z}_m] = \mathbb{P}_{\hat{\rho}}[\hat{X}_1] \dots \mathbb{P}_{\hat{\rho}}[\hat{Z}_m]$$

holds for all  $\hat{X}_1 \in \mathcal{L}(\mathcal{H}_1), \dots, \hat{Z}_m \in \mathcal{L}(\mathcal{H}_m)$ .

The commutative independence implies that the density operator  $\hat{\rho}_{\text{tot}} \in \mathcal{S}(\mathcal{H})$  is a tensor product state:  $\hat{\rho}_{\text{tot}} = \hat{\rho}_1 \otimes$

$\dots \otimes \hat{\rho}_m$ , where  $\hat{\rho} \in \mathcal{S}(\mathcal{H}_i)$  for  $i = 1, \dots, m$ . In particular, if  $\hat{\rho}_i = \hat{\rho}$  for all  $i = 1, \dots, m$ , the systems are called identically independent. If the commutative independence holds for quantum systems  $i = 1, \dots$ , the quantum version of the law of large number holds in the following sense (Batty (1979); Lindsay and Pata (1997)).

*Lemma 2.* (Quantum law of large numbers).

Let the Hilbert spaces be identical;  $\mathcal{H}_i = \mathcal{H}$  for  $i = 1, \dots, m$ . Let  $\hat{X}_i = \hat{X}_i^* \in \mathcal{L}(\mathcal{H}_i)$ ,  $i = 1, \dots$  be commutative independent variables, and  $\hat{\rho}_{\text{tot}} = \hat{\rho} \otimes \dots \hat{\rho}$ , where  $\hat{\rho} \in \mathcal{S}(\mathcal{H}_i)$ . Then

$$\mathbb{P}_{\hat{\rho}_{\text{tot}}} \left[ \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N f(\hat{X}_i) \right] = \mathbb{P}_{\hat{\rho}}[f(\hat{X})]$$

holds for any bounded function  $f$ .

There exists an operator  $\hat{Y} \in \mathcal{L}(\mathcal{H}_i)$  satisfying  $\hat{Y} = f(\hat{X}_i)$  for any bounded function  $f$  and  $\hat{X}_i$ , it is sufficient to consider the all linear bounded operators. Note that the commutative independence is essentially independence of classical random variables. There are other notions of independence in quantum probability theory (Muraki (2003)), however, we only focus on it and use Lemma 2.

## 2.2 Open quantum systems

As denoted above, any quantum system is described by suitable Hilbert space and linear operators on the Hilbert space. We consider two quantum systems, systems to be estimated/controlled and probed, respectively. The target system is denoted as  $\mathcal{H}_S$  and this composes  $N$  different identically same quantum subsystems;  $\mathcal{H}_i$ ,  $i = 1, \dots, N$ . The probed system is denoted as  $\mathcal{H}_P$  and also composes  $N$  different quantum fields.  $\mathcal{H}_P$  is a continuous Fock space (Guichardet (1969));  $\mathcal{H}_P = \otimes_{t \in [0, \infty)} \mathcal{H}_P(t)$ , where  $\mathcal{H}_P(t)$  is a Hilbert space at time  $t \geq 0$ . The compound quantum system is the tensor product Hilbert space  $\mathcal{H}_{\text{tot}} = \mathcal{H}_S \otimes \mathcal{H}_P$  equipped with a density operator  $\hat{\rho} = \hat{\rho}_S \otimes \hat{\rho}_P$ ,  $\hat{\rho}_S \in \mathcal{S}(\mathcal{H}_S)$ ,  $\hat{\rho}_P \in \mathcal{S}(\mathcal{H}_P)$ . For simplicity, we assume that  $\hat{\rho}_S$  consists of the tensor product state with the same small size density operator and  $\hat{\rho}_P$  is a vacuum state (Gardiner and Zoller (2004); Walls and Milburn (2008)). Physical quantities of the system are described by self-adjoint operators in  $\mathcal{L}(\mathcal{H}_S)$  and physical quantities of the probe system are described by self-adjoint operators in  $\mathcal{L}(\mathcal{H}_P)$ . They act on the total quantum system with the corresponding identity operator, though, we omit the identity operator for simplicity;  $\hat{X} \otimes \hat{1}_P \equiv \hat{X}$  and  $\hat{1}_S \otimes \hat{Y} \equiv \hat{Y}$  for  $\hat{X} \in \mathcal{L}(\mathcal{H}_S)$  and  $\hat{Y} \in \mathcal{L}(\mathcal{H}_P)$ .

According to quantum theory, the time evolution of every physical quantity  $\hat{X} = \hat{X}^* \in \mathcal{L}(\mathcal{H}_{\text{tot}})$  driven by probe system is determined by a unitary operator  $\hat{U}(t)$  which denotes the interaction between the system and the probe. We consider the unitary operator  $\hat{U}(t)$  as the solution of the following *Hudson–Parthasarathy equation*;

$$d\hat{U}(t) = \left( -i\hat{H}dt - \sum_{i=1}^N \left\{ \frac{1}{2} \hat{L}_i^* \hat{L}_i dt - \hat{L}_i d\hat{A}_i(t)^* + \hat{L}_i^* d\hat{A}_i(t) \right\} \right) \hat{U}(t) \quad (1)$$

with  $\hat{U}_0 = \hat{1}$ , where  $\hat{H} = \hat{H}^*$ ,  $\hat{L}_i \in \mathcal{L}(\mathcal{H}_S)$ , and  $\hat{A}_i(t) \in \mathcal{L}(\mathcal{H}_P)$  is a quantum annihilation process which satisfies quantum Ito's rule (Hudson and Parthasarathy (1984));

$$\begin{cases} d\hat{A}_i(t)d\hat{A}_j(t) = d\hat{A}_i(t)^*d\hat{A}_j(t) = d\hat{A}_i(t)dt = (dt)^2 = 0, \\ d\hat{A}_i(t)d\hat{A}_j(t)^* = \delta_{i,j}dt, \end{cases} \quad (2)$$

where  $\delta_{i,j}$  is the Kronecker's delta. Then the time evolution of the  $\hat{X}(t) = \hat{U}(t)^* \hat{X} \hat{U}(t)$  is given by the following quantum stochastic differential equation;

$$\begin{aligned} d\hat{X}(t) = & i[\hat{H}(t), \hat{X}(t)]_- dt \\ & + \frac{1}{2} \sum_{i=1}^N \left\{ \hat{L}_i^*(t)[\hat{X}(t), \hat{L}_i(t)]_- \right. \\ & \left. + [\hat{L}_i^*(t), \hat{X}(t)]_- \hat{L}_i(t) \right\} dt \\ & + \sum_{i=1}^N \left\{ [\hat{L}_i^*(t), \hat{X}(t)]_- d\hat{A}_i(t) \right. \\ & \left. + [\hat{X}(t), \hat{L}_i(t)]_- d\hat{A}_i^*(t) \right\} \end{aligned} \quad (3)$$

where  $\hat{H}(t) = \hat{U}(t)^* \hat{H} \hat{U}(t)$  and  $\hat{L}_i(t) = \hat{U}(t)^* \hat{L}_i \hat{U}(t)$ .

We consider the homodyne detection as a detection of the probe system (Gardiner and Zoller (2004); Walls and Milburn (2008)). The measurement outcomes are represented by  $\hat{Y}_i(t) := \hat{U}(t)^*(\hat{A}_i(t) + \hat{A}_i^*(t))\hat{U}(t)$ ,  $i = 1, \dots, N$  and its increment is

$$d\hat{Y}_i(t) = \left( \hat{L}_i(t) + \hat{L}_i^*(t) \right) dt + d\hat{A}_i(t) + d\hat{A}_i^*(t). \quad (4)$$

From the definitions of the unitary operator and the observed processes, following relations hold;

$$\hat{X}(t)\hat{Y}_i(s) = \hat{Y}_i(s)\hat{X}(t), \quad \forall t \geq s \geq 0, \quad (5)$$

$$\hat{Y}_i(t)\hat{Y}_j(s) = \hat{Y}_j(s)\hat{Y}_i(t), \quad \forall t, s \geq 0, \quad \forall i, j. \quad (6)$$

We denote the von Neumann subalgebra generated by  $\{\hat{Y}_i(s)\}_{s=0}^t$  by  $\mathcal{Y}_i(t)$ , which corresponds to the  $\sigma$ -field generated by measurement record up to time  $t$ . Clearly,  $\mathcal{Y}_i(t)$  is a commutative von Neumann subalgebra.

## 2.3 Quantum filter

Let  $\hat{\pi}_{i,t}(\hat{X}) := \mathbb{P}_{\hat{\rho}}[\hat{X}(t) | \mathcal{Y}_i(t)]$  be a quantum conditional expectation for  $i$ -th quantum system up to time  $t \geq 0$ . The quantum filtering equation is given by following equation (Bouten et al. (2007));

$$\begin{aligned} d\hat{\pi}_{i,t}(\hat{X}) = & \hat{\pi}_{i,t} \left( i[\hat{H}, \hat{X}]_- \right) dt \\ & + \frac{1}{2} \sum_{j=1}^N \hat{\pi}_{i,t} \left( \hat{L}_j^*[\hat{X}, \hat{L}_j]_- + [\hat{L}_j^*, \hat{X}]_- \hat{L}_j \right) dt \\ & + \hat{\pi}_{i,t} \left( (\hat{L}_i - \hat{\pi}_{i,t}(\hat{L}_i))^* \hat{X} \right. \\ & \left. + \hat{X}(\hat{L}_i - \hat{\pi}_{i,t}(\hat{L}_i)) \right) \\ & \times (d\hat{Y}_i(t) - \hat{\pi}_{i,t}(\hat{L}_i + \hat{L}_i^*)dt). \end{aligned} \quad (7)$$

Remember that  $\mathcal{Y}_t$  is identified as a set of classical random variables of the classical probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , there exists  $\hat{\rho}_t(\omega) \in \mathcal{S}(\mathcal{H}_S)$  for all  $\omega \in \Omega$  satisfies

$$\hat{\pi}_t(\hat{X})(\omega) = \text{Tr}[\hat{\rho}_t(\omega)\hat{X}], \quad \forall \hat{X} \in \mathcal{L}(\mathcal{H}_S), \quad \forall \omega \in \Omega.$$

By the cyclic property of the trace, the stochastic differential equation of  $\hat{\rho}_i(t)$ , so-called *the stochastic master equation* or *quantum trajectory equation*, is given by

$$\begin{aligned} d\hat{\rho}_i(t) = & -i \left[ \hat{H}, \hat{\rho}_i(t) \right]_- dt \\ & + \sum_{j=1}^N \left( \hat{L}_j \hat{\rho}_i(t) \hat{L}_j^* \right. \\ & \quad \left. - \frac{1}{2} \hat{L}_j^* \hat{L}_j \hat{\rho}_i(t) - \frac{1}{2} \hat{\rho}_i(t) \hat{L}_j^* \hat{L}_j \right) dt \\ & + \left( \hat{L}_i \hat{\rho}_i(t) + \hat{\rho}_i(t) \hat{L}_i^* - \text{Tr} \left[ (\hat{L}_i + \hat{L}_i^*) \hat{\rho}_i(t) \right] \hat{\rho}_i(t) \right) \\ & \quad \times \left( dy_i(t) - \text{Tr} \left[ (\hat{L}_i + \hat{L}_i^*) \hat{\rho}_i(t) \right] dt \right). \end{aligned} \quad (8)$$

The stochastic master equation is a quantum version of a Kushner–Stratonovich equation (Bouten et al. (2007)). When the optimal feedback control problem is considered, it is solved by the HJB equation together with Kushner–Stratonovich equation (Bensoussan (1992)). Similarly, together with the stochastic master equation, we can obtain a quantum version of the HJB equation (Bouten et al. (2005); Gough et al. (2005); Bouten et al. (2009); Belavkin et al. (2009)). However, if the system consists of  $N$  qubits, it requires  $2^N$ -dimensional matrices to calculate Eq. (8), which is impractical if  $N$  is large. In order to overcome this problem, we develop a quantum version of mean-field type control problem in the following section.

### 3. A QUANTUM MEAN-FIELD TYPE FILTERING

#### 3.1 Mean-field approximation of the quantum system

If the Hamiltonian contains weak interaction among many quantum systems, there are some methods to approximate the whole system. Especially, the mean-field approximation of quantum spin systems described by the Ising model is a very popular method to analyze when phase transition occurs (Schulz (1996); Sachdev and Bhatt (1990); Son and Vedral (2018)). Consider the following Ising type Hamiltonian with control inputs;

$$\hat{H} = \sum_{i=1}^N u_i(t) \hat{H}_i + \alpha \sum_{i,j=1, i \neq j}^N \hat{Z}_i \hat{Z}_j, \quad (9)$$

where  $\hat{H}_i = \hat{H}_i^* \in \mathcal{L}(\mathcal{H}_i)$  is the Hamiltonian of  $i$ -th quantum system and  $\hat{Z}_i \hat{Z}_j$ , where  $\hat{Z}_i = \hat{Z}_i^* \in \mathcal{L}(\mathcal{H}_i)$ , means the interaction between  $i$ -th and  $j$ -th quantum systems.  $u_i(t)$  is control input for  $i$ -th system and  $\alpha > 0$  is an interaction coefficient. If  $\alpha$  is small enough, we replace  $\alpha$  with  $\beta/N$  and then the dynamics (3) for  $i$ -th quantum system becomes

$$\begin{aligned} d\hat{X}_i^{(N)}(t) = & u_i(t) i [\hat{H}_i(t), \hat{X}_i^{(N)}(t)]_- dt \\ & + i\beta [\hat{Z}_i(t), \hat{X}_i^{(N)}(t)]_- \left( \frac{1}{N} \sum_{j=1, j \neq i}^N \hat{Z}_j(t) \right) \\ & + \frac{1}{2} \left( \hat{L}_i^*(t) [\hat{X}_i^{(N)}(t), \hat{L}_i(t)]_- \right. \\ & \quad \left. + [\hat{L}_i^*(t), \hat{X}_i^{(N)}(t)]_- \hat{L}_i(t) \right) dt \\ & + [\hat{L}_i^*(t), \hat{X}_i^{(N)}(t)]_- d\hat{A}_i(t) \\ & + [\hat{X}_i^{(N)}(t), \hat{L}_i(t)]_- d\hat{A}_i^*(t), \end{aligned} \quad (10)$$

where  $X_i^{(N)}$  is a bounded operator in  $\mathcal{L}(\mathcal{H}_i)$ . In order to develop a quantum version of mean-field type control problem, it is necessary to deal with the empirical mean

$$\frac{1}{N} \sum_{j=1, j \neq i}^N \hat{Z}_j(t) \quad (11)$$

in a suitable manner.

*Proposition 3.* For any  $i$  and  $j$ ,

$$\lim_{N \rightarrow \infty} \left| \mathbb{P}_{\hat{\rho}} [\hat{X}_i^{(N)}(t) \hat{X}_j^{(N)}(t)] - \mathbb{P}_{\hat{\rho}} [\hat{X}_i^{(N)}(t)] \mathbb{P}_{\hat{\rho}} [\hat{X}_j^{(N)}(t)] \right| = 0$$

for any linear bounded operator  $\hat{X}_i^{(N)} \in \mathcal{L}(\mathcal{H}_i)$  and  $\hat{X}_j^{(N)} \in \mathcal{L}(\mathcal{H}_j)$ .

This proposition indicates that the asymptotically commutative independence holds for (10).

**Proof.** We employ the fundamental theorem of calculus. For any adequate size operator  $\hat{X}$ ,  $\hat{H}$ ,  $\hat{L}$ , and  $d\hat{A}$ , we introduce the following operator.

$$\begin{aligned} \mathcal{D}_u(\hat{X}, \hat{H}, \hat{L}, d\hat{A}) \\ := & i[\hat{H}, \hat{X}]_- dt + \frac{1}{2} \left( \hat{L}^* [\hat{X}, \hat{L}]_- + [\hat{L}^*, \hat{X}]_- \hat{L} \right) dt \\ & + [\hat{L}^*, \hat{X}]_- d\hat{A} + [\hat{X}, \hat{L}]_- d\hat{A}^* \end{aligned}$$

Then, by using quantum Ito calculus,

$$\begin{aligned} & \frac{d}{dt} \mathbb{P}_{\hat{\rho}} \left[ \hat{X}_i^{(N)}(t) \hat{X}_j^{(N)}(t) \right] \\ = & \mathbb{P}_{\hat{\rho}} \left[ \mathcal{D}_{u_i(t)}(\hat{X}_i^{(N)}(t), \hat{H}_i(t), \hat{L}_i(t), d\hat{A}_i(t)) \hat{X}_j^{(N)}(t) \right] \\ & + \mathbb{P}_{\hat{\rho}} \left[ \hat{X}_i^{(N)}(t) \mathcal{D}_{u_j(t)}(\hat{X}_j^{(N)}(t), \hat{H}_j(t), \hat{L}_j(t), d\hat{A}_j(t)) \right] \\ & + O(N^{-1}) \end{aligned}$$

and

$$\begin{aligned} & \frac{d}{dt} \mathbb{P}_{\hat{\rho}} \left[ \hat{X}_i^{(N)}(t) \right] \mathbb{P}_{\hat{\rho}} \left[ \hat{X}_j^{(N)}(t) \right] \\ = & \mathbb{P}_{\hat{\rho}} \left[ \mathcal{D}_{u_i(t)}(\hat{X}_i^{(N)}(t), \hat{H}_i(t), \hat{L}_i(t), d\hat{A}_i(t)) \right] \mathbb{P}_{\hat{\rho}} \left[ \hat{X}_j^{(N)}(t) \right] \\ & + \mathbb{P}_{\hat{\rho}} \left[ \hat{X}_i^{(N)}(t) \right] \\ & \quad \times \mathbb{P}_{\hat{\rho}} \left[ \mathcal{D}_{u_j(t)}(\hat{X}_j^{(N)}(t), \hat{H}_j(t), \hat{L}_j(t), d\hat{A}_j(t)) \right] \\ & + O(N^{-1}) \end{aligned}$$

are obtained. Note that the cross term between  $i$  and  $j$  is included in  $O(N^{-1})$ . Since the initial state is a tensor product state,

$$\begin{aligned} & \frac{d}{dt} \mathbb{P}_{\hat{\rho}} \left[ \hat{X}_i^{(N)}(0) \hat{X}_j'^{(N)}(0) \right] \\ = & \mathbb{P}_{\hat{\rho}} \left[ \mathcal{D}_{u_i(0)}(\hat{X}_i^{(N)}(0), \hat{H}_i(0), \hat{L}_i(0), d\hat{A}_i(0)) \right] \mathbb{P}_{\hat{\rho}} \left[ \hat{X}_j'^{(N)}(0) \right] \\ & + \mathbb{P}_{\hat{\rho}} \left[ \hat{X}_i^{(N)}(0) \right] \\ & \times \mathbb{P}_{\hat{\rho}} \left[ \mathcal{D}_{u_j(0)}(\hat{X}_j'^{(N)}(0), \hat{H}_j(0), \hat{L}_j(0), d\hat{A}_j(0)) \right] \\ & + O(N^{-1}) \\ = & \frac{d}{dt} \mathbb{P}_{\hat{\rho}} \left[ \hat{X}_i^{(N)}(0) \right] \mathbb{P}_{\hat{\rho}} \left[ \hat{X}_j'^{(N)}(0) \right]. \end{aligned}$$

Therefore, as  $N \rightarrow \infty$ , the cross term vanishes and the equations become the same with the same initial condition. This implies the statement of the proposition holds.

From above, the mean-field approximation probably works well if  $N$  is large. We need to describe the approximate of Eq. (11) and it is shown in the following subsection.

### 3.2 Mean-field type quantum filter

We denote  $\hat{X}_i(t) = \lim_{N \rightarrow \infty} \hat{X}_i^{(N)}(t)$  for convenience. Since  $\mathcal{Y}_i(t)$  is the information from the  $i$ -th output, for  $j \neq i$ ,  $\hat{\pi}_{i,t}(\hat{X}_j^{(N)}) = \mathbb{P}_{\hat{\rho}}[\hat{X}_j^{(N)}(t) | \mathcal{Y}_i(t)] \rightarrow \mathbb{P}_{\hat{\rho}}[\hat{X}_j(t)]$  as  $N \rightarrow \infty$ . The quantum master equation of the  $j$ -th system is then

$$\begin{aligned} \frac{d}{dt} \tilde{\rho}(t) = & -u(t) i[\hat{H}, \tilde{\rho}(t)]_- - i[\hat{Z}, \tilde{\rho}(t)]_- \text{Tr}[\tilde{\rho}(t) \hat{Z}] \\ & + \frac{1}{2} \left( 2\hat{L} \tilde{\rho}(t) \hat{L}^* - \hat{L}^* \hat{L} \tilde{\rho}(t) - \tilde{\rho}(t) \hat{L}^* \hat{L} \right). \quad (12) \end{aligned}$$

We call Eq. (12) *mean-field(MF) type quantum master equation* and this is the quantum counterpart of classical mean-field Fokker–Planck equation. The original system's dimension is  $\dim(\mathcal{H}_j)^N$ , however, this nonlinear quantum master equation requires  $\dim(\mathcal{H})$ -dimensional calculation. Remember  $\mathcal{H}_i = \mathcal{H}$  for any  $i = 1, \dots, N$ .

Together with the solution of Eq. (12), the mean-field type quantum filter and the mean-field type stochastic master equation is derived as follows.

**Theorem 4.** (MF type quantum filter). Let  $\tilde{\pi}_{i,t}(\hat{X})$  be the limit of  $\hat{\pi}_{i,t}(\hat{X}_i^{(N)})$  for  $\hat{X}_i^{(N)} \in \mathcal{L}(\mathcal{H}_i)$ . For any  $i$ -th system and  $\hat{X} \in \mathcal{L}(\mathcal{H}_i)$ ,

$$\begin{aligned} d\tilde{\pi}_{i,t}(\hat{X}) = & \tilde{\pi}_{i,t} \left( u_i(t) i[\hat{H}, \hat{X}]_- \right) dt \\ & + \tilde{\pi}_{i,t} \left( i[\hat{Z}, \hat{X}]_- \right) \text{Tr} \left[ \tilde{\rho} \hat{Z} \right] dt \\ & + \frac{1}{2} \tilde{\pi}_{i,t} \left( \hat{L}^* [\hat{X}, \hat{L}]_- + [\hat{L}^*, \hat{X}]_- \hat{L} \right) dt \\ & + \tilde{\pi}_{i,t} \left( (\hat{L} - \tilde{\pi}_{i,t}(\hat{L}))^* \hat{X} \right. \\ & \quad \left. + \tilde{X}(\hat{L} - \tilde{\pi}_{i,t}(\hat{L})) \right) \\ & \times (d\tilde{Y}_i(t) - \tilde{\pi}_{i,t}(\hat{L} + \hat{L}^*) dt) \end{aligned}$$

**Proof.** Applying the general quantum filter to Eq. (10) results

$$\begin{aligned} d\hat{\pi}_{i,t}(\hat{X}_i^{(N)}) = & \hat{\pi}_{i,t} \left( u_i(t) i[\hat{H}_i, \hat{X}_i^{(N)}]_- \right) dt \\ & + \hat{\pi}_{i,t} \left( i[\hat{Z}_i, \hat{X}_i^{(N)}]_- - \frac{1}{N} \sum_{j=1, j \neq i}^N \hat{Z}_j \right) dt \\ & + \frac{1}{2} \hat{\pi}_{i,t} \left( \hat{L}_i^* [\hat{X}_i^{(N)}, \hat{L}_i]_- + [\hat{L}_i^*, \hat{X}_i^{(N)}]_- \hat{L}_i \right) dt \\ & + \hat{\pi}_{i,t} \left( (\hat{L}_i - \hat{\pi}_{i,t}(\hat{L}_i))^* \hat{X}_i^{(N)} \right. \\ & \quad \left. + \hat{X}_i^{(N)}(\hat{L}_i - \hat{\pi}_{i,t}(\hat{L}_i)) \right) \\ & \times (d\hat{Y}_i(t) - \hat{\pi}_{i,t}(\hat{L}_i + \hat{L}_i^*) dt). \end{aligned}$$

Since

$$\begin{aligned} \lim_{N \rightarrow \infty} \hat{\pi}_{i,t} \left( i[\hat{Z}_i, \hat{X}_i^{(N)}]_- - \frac{1}{N} \sum_{j=1, j \neq i}^N \hat{Z}_j \right) \\ = \tilde{\pi}_{i,t} \left( i[\hat{Z}, \hat{X}]_- \right) \text{Tr} \left[ \tilde{\rho} \hat{Z} \right] \end{aligned}$$

and the conditional expectation is then only the functional on  $\mathcal{L}(\mathcal{H}_i)$ , we omit the subscript and obtain the mean-field type quantum filter. Then the statement of the theorem holds.

From the same argument of quantum filter,  $\tilde{\rho}_i(t) \in \mathcal{S}(\mathcal{H})$  satisfying

$$\tilde{\pi}_{i,t}(\hat{X}) = \text{Tr}[\tilde{\rho}_i(t) \hat{X}], \quad \forall \hat{X} \in \mathcal{L}(\mathcal{H}).$$

The time evolution of  $\tilde{\rho}_i(t)$  is then given by the following proposition.

**Proposition 5.** (MF type stochastic master equation). For any  $i$ -th system,

$$\begin{aligned} d\tilde{\rho}_i(t) = & -u(t) i[\hat{H}, \tilde{\rho}_i(t)]_- - i[\hat{Z}, \tilde{\rho}_i(t)]_- \text{Tr}[\tilde{\rho}_i(t) \hat{Z}] \\ & + \frac{1}{2} \left( 2\hat{L} \tilde{\rho}_i(t) \hat{L}^* - \hat{L}^* \hat{L} \tilde{\rho}_i(t) - \tilde{\rho}_i(t) \hat{L}^* \hat{L} \right) \\ & + \left( \hat{L} \tilde{\rho}_i(t) + \tilde{\rho}_i(t) \hat{L}^* - \text{Tr} \left[ (\hat{L} + \hat{L}^*) \tilde{\rho}_i(t) \right] \tilde{\rho}_i(t) \right) \\ & \times \left( dy_i(t) - \text{Tr} \left[ (\hat{L} + \hat{L}^*) \tilde{\rho}_i(t) \right] dt \right). \quad (13) \end{aligned}$$

Since it is the same as the derivation of the stochastic master equation, we omit the proof.

In order to develop a quantum version of mean-field games or mean-field type control problems, it is necessary to define the objective function and derive the HJB equation together with Eqs. (12) and (13). The derivation of the HJB equation is the same as Bouten et al. (2005); Gough et al. (2005); Bouten et al. (2009); Belavkin et al. (2009), we also omit the mean-field type HJB equation. Hence we can formulate a quantum version of the mean-field type control problem.

## 4. CONCLUSION

In this paper, we developed a mean-field type quantum filter for Ising type open quantum systems. Due to the page limitation, we omit to describe the mean-field type control problem explicitly. A mean-field type quantum master equation, quantum filter and stochastic master equation are derived. We do not mention control input, however, this is future work. We will also evaluate solutions to the quantum mean-field type control problems.

Finally, I mention a paper Kolokoltsov (2020) published on a preprint server recently. Kolokoltsov (2020) also

deals with quantum mean-field games and derives a mean-field type quantum master equation and stochastic master equation, which are essentially the same as Eqs. (12) and (13). Besides, he shows that the solutions of the quantum mean-field game are shown to specify approximate Nash equilibria. However, there are some differences from our approach. For example, they consider a specific interaction operator  $\hat{L}^* = -\hat{L}$ . He uses a quantum version of empirical measures, however, we avoid using them.

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