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Conditions of Almost Sure Boundedness and Practical Asymptotic Stability of Continuous-Time Stochastic Systems ⋆

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Abstract:
This paper investigates the boundedness conditions of solutions of stochastic differential equations in the almost sure sense. Boundedness is one of the most fundamental properties in a lot of control problems. In general, it is hard to investigate almost sure boundedness of solutions of stochastic differential equations, unlike deterministic systems. However, for a class of systems, the almost sure boundedness can be investigated. This paper deals with conditions for the almost sure boundedness of stochastic systems, which is based on boundary properties of one-dimensional diffusion processes. Moreover, based on the boundedness, we show the characterization of a kind of practical asymptotic stability of one-dimensional stochastic systems in the almost sure sense. The presented results are validated through a numerical example.

Keywords: Stochastic Systems, Stochastic Control, Nonlinear Systems, Stability Analysis, Lyapunov methods

1. INTRODUCTION

One of the fundamental properties in control theory is boundedness. In particular, for systems affected by disturbances, it is essential to develop tools for investigating the boundedness. The boundedness of deterministic systems can be characterized by Lyapunov-type conditions (see (Khalil, 2002, Chapter 4), for example). Moreover, some recent studies focus on barrier functions, which are utilized to investigate the boundedness, especially in safety verification (for example, see Wieland and Allgöwer (2007); Ames et al. (2016)).

When dynamical systems are exposed to stochastic disturbances, such systems are usually modeled by using stochastic differential equations. The class of such systems has been extensively studied, which includes studies on optimal control by Krylov (2008); Fleming and Soner (2006); Yong and Zhou (1999), and stability and stabilization of stochastic systems by Khasminskii (2012); Kushner (1967); Mao (2007). In particular, as seen in Khasminskii (2012); Kushner (1967); Mao (2007), stochastic Lyapunov theory is a useful tool to investigate asymptotic behaviors, such as notions of stability and asymptotic stability. In the analysis of the stability of stochastic systems, it is assumed that systems possess an almost sure equilibrium. That is, the diffusion coefficients of systems vanish at the equilibrium. As for non-vanishing diffusion coefficient cases, it is required to show relations between the effects of stochastic noises and asymptotic behaviors of stochastic systems, which include the boundedness.

Boundedness notions of stochastic systems have been developed by some authors, and several types of the boundedness can be defined, for example, almost sure sense, the sense of moments, and the sense in probability. In Zakai (1967) and Miyahara (1972), the boundedness in the sense of moments has been investigated. Moreover, stochastic versions of input-to-state stability also have been studied in Deng et al. (2001); Tsinias (1998); Ito and Nishimura (2015). In terms of the almost sure sense, it might be possible to investigate almost sure boundedness with some results of almost sure stability in Bardi and Cesaroni (2005); Nishimura (2016). Those results allow handling systems that possess non-vanishing diffusion terms in various ways. However, it is still in progress to establish the boundedness, especially in the almost sure sense.

This study focuses on properties of one-dimensional diffusion processes as tools to discuss the boundedness in the almost sure sense. For one-dimensional diffusion processes, behaviors of the stochastic processes at boundaries of given intervals have been extensively studied by Feller (1954); Itô and McKean (2012). Feller (1954) shows a classification of boundaries on which the one-dimensional diffusions are defined, such as regular, exit, natural, and entrance boundaries, with notions of speed measures and scale functions. Such a classification allows characterizing whether or not one-dimensional stochastic processes can reach boundaries. Moreover, conditions have been estab-

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lished for characterizing the boundaries, which can be found in Itô and McKean (2012); Rogers and Williams (2000); Revuz and Yor (2005). Based on those characterizations, Khasminskii (2012) shows conditions of the finite-time explosion of solutions of stochastic differential equations. To develop a method for handling the boundedness of stochastic systems, employing such characterizations of boundaries is one of the feasible approaches.

This paper provides conditions of the boundedness of solutions of stochastic differential equations in the almost sure sense. Those conditions are shown by utilizing the classifications of boundaries of one-dimensional diffusion processes and their properties. In the case of one-dimensional systems, a condition of the boundedness is a straightforward extension of the results on the boundaries of the one-dimensional diffusion processes. Moreover, this paper investigates a condition of a kind of almost sure practical stability based on the results for the boundedness. In the case of general n-dimensional systems, a condition of the boundedness is provided by using Lyapunov-like functions. In addition, this paper provides an example with the almost sure boundedness and practical asymptotic stability.

The rest of this paper is constructed as follows. In the following section, we provide examples to motivate discussions on the boundedness of solutions of stochastic differential equations. Then, we provide preliminaries on the stochastic differential equations. Following the preliminaries, we provide a condition of boundedness for one-dimensional systems based on the boundary classifications and their properties. The section also provides results on practical stability. Then, we provide conditions of the boundedness of n-dimensional systems. Finally, we show an example of bounded systems, and we give conclusions.

Notations: The set of real numbers is denoted as $\mathbb{R}$, the set of nonnegative real numbers is denoted as $\mathbb{R}_{\geq 0}$, and the n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. For real numbers $a, b \in \mathbb{R}$, (a, b), $[a, b]$ denote an open interval and a closed interval, respectively. For a sufficiently continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, $(\partial V)/(\partial x)(x)$, $(\partial^2 V)/(\partial x^2)(x)$ denote the gradient and Hessian of $V(x)$, respectively. To consider stochastic systems, a filtered probability space is denoted by $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{P})$, where Ω is the sample space, $\mathcal{F}$ is a σ-algebra of Ω, $\mathcal{F}_t$ is a filtration of $\mathcal{F}$, and P is a probability measure on $\mathcal{F}$. $\mathrm{E}[X]$ denotes an expectation of a random variable X.

2. MOTIVATIONAL EXAMPLE

This section provides motivational examples of stochastic systems to discuss boundedness and practical asymptotic stability. The later sections provide conditions to characterize the boundedness and practical asymptotic stability of the presented examples.

First, let us consider the following one-dimensional stochastic system, which is given by

$$dX_t = -c_1 X_t dt + c_2 dW_t, \quad X_0 = x, \tag{1}$$

where $X_t \in \mathbb{R}$ is the state, W_t is the one-dimensional standard Wiener process, c_1 and c_2 are constants such that $c_1 > 0$ and $c_2 \neq 0$, and $x \in \mathbb{R}$ is the initial value of the state. If $c_2 = 0$, the system (1) reduces to an asymptotically stable deterministic system, and the state X_t converges to the origin. When $c_2 \neq 0$, one might expect intuitively that the state of (1) would converge to a bounded neighborhood of the origin because the drift term of (1) drives the state toward the origin. However, when $t \to \infty$, the probability density p_∞ of $X_\infty = \lim_{t\to\infty} X_t$ is given by

$$p_\infty(X_\infty) = \sqrt{\frac{c_1}{2\pi c_2^2}} \exp\left(-\frac{c_1 X_\infty^2}{2c_2^2}\right). \tag{2}$$

This probability density (2) implies that the state X_∞ exists in a region $\{X \in \mathbb{R} \mid |X| > R\}$ with non-zero probability for any $R > 0$.

Then, we look into another example of one-dimensional stochastic systems, which is given as

$$dX_t = (1 - 2X_t)\, dt + |X_t(1 - X_t)|^{1/2}\, dW_t, \quad X_0 = x, \tag{3}$$

where X_t, W_t, and x are the same as those of (1). Please note that its drift term is linear, but the diffusion coefficient is nonlinear and also non-Lipschitz continuous. This example is inspired by Example 8 on p. 239 of Karlin and Taylor (1981). Please note that the drift term $(1-2x)$ of (3) vanishes at $x = 0.5$ and the nonlinear diffusion coefficient $|x(1 - x)|^{1/2}$ vanishes at $x = 0$ and $x = 1$. Therefore, the system (3) does not have any equilibria. This implies that we cannot consider the asymptotic stability of an equilibrium to investigate asymptotic behaviors of this system. We will consider the boundedness of this system as an asymptotic property of the system (3). Fig. 1 shows ten sample paths of the state of (3) with $x = 0.5$, which are obtained by using the Euler–Maruyama method. Fig. 1 shows that all the sample paths are bounded in $[0, 1]$ for $t \geq 0$. This behavior differs from that of the system (1). As seen in (2), the state of the system (1) can reach any region far from the origin. On the other hand, the system (3) stays in a bounded region. This paper provides a characterization to capture such behaviors shown in Fig. 1.

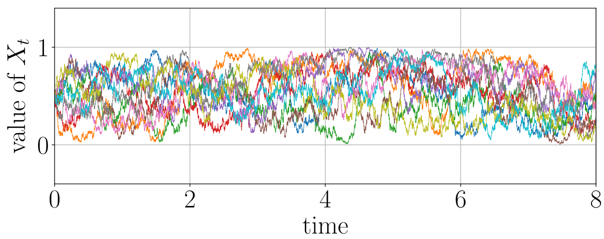

Fig. 1. Sample Paths of Time Responses of System (3) with $x = 0.5$

3. PRELIMINARIES

This section gives descriptions of stochastic systems given by stochastic differential equations. Throughout this section, we consider an Itô stochastic differential equation

$$dX_t = f(X_t)dt + \sigma(X_t)dW_t, \ X_0 = x \in \mathbb{R}^n \tag{4}$$

where $X_t \in \mathbb{R}^n$ is the state, W_t is a one-dimensional standard Wiener process, $f : \mathbb{R}^n \to \mathbb{R}^n$, and $\sigma : \mathbb{R}^n \to \mathbb{R}^n$. The initial value of the state is given by $X_0 = x$. It is supposed that $f(x)$ and $\sigma(x)$ are continuous in x, which implies that systems can be non-Lipschitz continuous as in the example (3). Therefore, this paper employs a solution concept called a weak solution of stochastic differential equations, which is suitable for systems that are not necessarily Lipschitz continuous. The definition is given as follows, which is primarily taken from Definition 2.1 in Chapter IV of Ikeda and Watanabe (1989) with minor modifications (see also Chapter IX, §1 of Revuz and Yor (2005) and Karatzas and Shreve (2012)).

Definition 1. (Solutions of Stochastic Differential Equations). A pair (X_t, W_t) of stochastic processes X_t and W_t is said to be a solution of the stochastic differential equation (4) if there exists a filtered probability space $(\Omega, \mathcal{F}, \mathrm{P}, \mathcal{F}_t)$ such that the stochastic process W_t is an $\mathcal{F}_t$-adapted Wiener process, the stochastic process X_t is a continuous $\mathcal{F}_t$-adapted stochastic process, and X_t and W_t satisfy

$$X_t - X_0 = \int_0^t f(X_s)ds + \int_0^t \sigma(X_s)dW_s, \quad 0 \le t < \tau_\infty$$

with probability one, where τ_∞ is given by

$$\tau_\infty = \inf\{t \in [0,\infty) \mid \|x_t\| = \infty\}.$$

Given Definition 1, the following result guarantees the existence of a solution under the condition of the continuity of $f(x)$ and $\sigma(x)$ (Ikeda and Watanabe, 1989, Theorem 2.3 of Chapter IV).

Theorem 1. (Ikeda and Watanabe (1989)). Suppose that $f(x)$ and $\sigma(x)$ of (4) are continuous on $\mathbb{R}^n$. Then there exists a solution of the stochastic differential equation (4) in the sense of Definition 1.

4. BOUNDEDNESS AND PRACTICAL ASYMPTOTIC STABILITY OF ONE-DIMENSIONAL STOCHASTIC SYSTEMS

This section investigates the boundedness and the practical asymptotic stability of one-dimensional stochastic systems. Main tools are scale functions, speed measures, and related notions of one-dimensional diffusion processes, which can be found in Rogers and Williams (2000); Itô and McKean (2012); Revuz and Yor (2005); Kallenberg (2006); Karlin and Taylor (1981). Therefore, this section first introduces such related notions. Then, by using the scale functions and speed measures, we investigate the boundedness and the practical asymptotic stability.

4.1 Scale Functions, Speed Measures, and Related Notions and Results

The following discussion considers a one-dimensional stochastic differential equation given by

$$dX_t = a(X_t)dt + b(X_t)dW_t, \quad X_0 = x \in \mathbb{R}, \tag{5}$$

where $X_t \in \mathbb{R}$ is the state variable, x is the initial value of the state, W_t is a one-dimensional standard Wiener process, $a : \mathbb{R} \to \mathbb{R}$, and $b : \mathbb{R} \to \mathbb{R}$. We assume that $a(x)$ and $b(x)$ are continuous on $\mathbb{R}$. The infinitesimal generator of the system (5) is given by

$$\mathcal{L} = a(x)\frac{\partial}{\partial x} + \frac{1}{2}b(x)^2\frac{\partial^2}{\partial x^2}. \tag{6}$$

In the following, we consider the behavior of the system (5) on an open interval $I = (l, r) \subset \mathbb{R}$, and we assume that $b(x) \neq 0$ on I.

We study the boundedness of the one-dimensional stochastic systems by investigating a condition whether the state X_t reaches the boundaries of I. By investigating the condition, we can discuss the boundedness of the solution of (5). To this end, consider sequences l_n and r_n $(n = 1, 2, \dots)$ such that $l_n \downarrow l$, $r_n \uparrow r$, and let stopping times τ_{l_n} and τ_{r_n} be given by

$$\tau_{l_n} = \inf\{t > 0; X_t = l_n\}, \tag{7}$$

$$\tau_{r_n} = \inf\{t > 0; X_t = r_n\}. \tag{8}$$

Then, with the stopping times $\tau_l = \lim_{n\to\infty} \tau_{l_n}$ and $\tau_r = \lim_{n\to\infty} \tau_{r_n}$,

$$\lim_{t\uparrow\tau_l} X_t = l, \quad \lim_{t\uparrow\tau_r} X_t = r \tag{9}$$

hold, respectively. With the stopping times τ_l and τ_r, the behavior of (5) can be studied by evaluating

$$\mathrm{P}\{\tau_l < \infty\} \text{ and } \mathrm{P}\{\tau_r < \infty\}, \tag{10}$$

which give the probability that the state X_t reaches the boundaries l and r in finite time.

To evaluate the probabilities, we introduce a scale function and a speed measure. Consider

$$s(x) = \int_c^x \exp\left(-\int_c^y \frac{2a(z)}{b(z)^2}dz\right)dy, \tag{11}$$

where c is the base point of the integration, which can be arbitrarily chosen from the interval I. The function $s(x)$ of (11) is called the scale function of (5). Moreover, we introduce a function given by

$$m'(x) = \frac{2}{b(x)^2 s'(x)}. \tag{12}$$

With $b_1, b_2 \in I$, the measure m given by

$$m([b_1, b_2]) = \int_{b_1}^{b_2} m'(y)dy \tag{13}$$

is called a speed measure. With the scale function $s(x)$ of (11) and the speed measure $m([b_1, b_2])$ of (13), we can investigate behaviors of (5). For example, it is possible to investigate whether X_t reaches the boundary of the interval $I = (l, r)$ in finite time. To investigate such properties, we introduce further notions, which are given as functions $\nu, \mu : I \to \mathbb{R}$ as

$$\nu(x) = \int_c^x m([c, y])ds'(y) \tag{14}$$

$$\mu(x) = \int_c^x s(y)dm(y) \tag{15}$$

with $c \in I$, which is the same as in (11). Then, the following definition of a boundary is introduced, which can be found in Karlin and Taylor (1981); Itô and McKean (2012); Rogers and Williams (2000); Revuz and Yor (2005).

Definition 2. The boundary l of I is said to be an entrance boundary if $\nu(l+) = \infty$, $\mu(l+) < \infty$. Similarly, the boundary r is said to be an entrance boundary if $\nu(r-) = \infty$, $\mu(r-) < \infty$.

This study focuses on the entrance boundary. For the entrance boundary, the following holds, which can be found in Karlin and Taylor (1981) with minor modifications.

Theorem 2. (Karlin and Taylor (1981)). Consider the system (5) and assume that the boundaries of I, l and r are entrance boundaries. Then, $\mathbb{P}\{\tau_l = \infty\} = 1$ and $\mathbb{P}\{\tau_r = \infty\} = 1$ hold.

Therefore, when $\nu(l+) = \infty$, $\nu(r-) = \infty$ hold and the initial value x satisfies $x \in I$, the state never reaches the boundary $x = l$ or $x = r$ in finite time. Moreover, the condition $\mu(l+) < \infty$ of the entrance boundary implies that when the initial values x of X_t is given as $x = l$, X_t reaches in the interior of I (Karlin and Taylor (1981)). By using the notion of the boundaries, we develop conditions of boundedness of stochastic systems.

4.2 Condition of Boundedness of One-Dimensional Systems

This subsection provides the boundedness of one-dimensional stochastic systems. The condition is a straightforward extension of the results introduced in the previous subsection.

The following theorem gives a condition of the boundedness of the stochastic system (5).

Theorem 3. Let x be the initial value of the system (5), which satisfies $x \in I$. Suppose that $b(x) \neq 0$ on I and that

$$\nu(l+) = \infty,\ \nu(r-) = \infty,\ \mu(l+) < \infty,\ \mu(r-) < \infty \quad (16)$$

hold. Then,

$$\mathrm{P}\left\{X_t \in I \text{ for } t \geq 0\right\} = 1 \quad (17)$$

holds.

Proof. Because we assume that $a(x)$ and $b(x)$ are continuous on $\mathbb{R}$, Theorem 1 implies the local existence of the solution of (5) in time with the initial value $X_0 = x \in I$. Then, according to the conditions in (16) and Theorem 2, we obtain

$$\mathrm{P}\left\{\tau_l = \infty\right\} = 1, \quad \mathrm{P}\left\{\tau_r = \infty\right\} = 1. \quad (18)$$

The equations in (18) imply that the state X_t never reach the boundaries of the set I in finite time. Moreover, the conditions in (16) imply the boundaries l, r is entrance boundaries. Therefore, the solution stays in I for $t \geq 0$, and the solution exists for $t \geq 0$. This implies that (17) holds. This completes the proof.

4.3 Practical Asymptotic Stability of One-Dimensional Systems

Based on the boundedness results in the preceding section, we investigate a condition of a kind of practical stability of one-dimensional stochastic systems. The result presented here is a Lyapunov-like condition.

The following result provides a condition of the practical asymptotic stability.

Theorem 4. Consider the system (5). Suppose that there exist intervals $I = (l, r)$, $U \subset \mathbb{R}$ with $U \subset I$, and a function $V : \mathbb{R} \to \mathbb{R}$ such that the followings hold:

(CO1) For the coefficient functions a and b of the system (5), there exists a constant $K > 0$ such that for any $x \in \mathbb{R}$,

$$|a(x)|^2 + |b(x)|^2 \leq K\left(1 + |x|^2\right) \quad (19)$$

holds.

(CO2) $V(x)$ is a twicely continuously differentiable function, nonnegative, radially unbounded, and there exists $k > 0$ such that

$$\mathcal{L}V(x) < -k \text{ for } x \in \mathbb{R} \setminus U \quad (20)$$

holds where $\mathcal{L}$ is the infinitesimal generator of the system (5), which is given in (6),

(CO3) the conditions

$$\begin{aligned} \nu(l+) = \infty, \quad \mu(l+) < \infty, \\ \nu(r-) = \infty, \quad \mu(r-) < \infty \end{aligned} \quad (21)$$

hold where the functions μ and ν are given in (14) and (15).

Then, when $x \notin I$,

(RO1) with $\tau_I = \inf\left\{t > 0 \mid X_t \in I\right\}$,

$$\mathrm{P}\left\{\tau_I < \infty\right\} = 1. \quad (22)$$

(RO2) Moreover,

$$\mathrm{P}\left\{X_t \in I \text{ for } t \geq \tau_I\right\} = 1 \quad (23)$$

holds.

When $x \in I$,

$$\mathrm{P}\left\{X_t \in I \text{ for } t \geq 0\right\} = 1. \quad (24)$$

holds.

Remark 1. Please note that we do not assume the existence of an almost sure equilibrium for the system (5) in the above theorem. Theorem 4 is a condition of a kind of practical asymptotic stability of stochastic systems without equilibria. The property (RO1) means that X_t reaches the interval I from the outside the interval in finite time with probability one. The property (RO2) means that X_t never leaves the interval I with probability one after it enters the interval I. Roughly speaking, the interval I is a positive invariant set in the almost sure sense and the interval is also an attractive set.

Proof. We first show the existence of the solutions of the system. Then, we show that when $x \notin I$, the state reaches the boundary of the set I in finite time with probability one, and the state never leaves the set I in finite time with probability one.

The global existence of the solution of the system (5) is guaranteed by the condition (CO1) and Theorem 2.2 on p. 177 of Ikeda and Watanabe (1989).

Then, we show that the stopping time τ_I in (22) is finite with probability one. Let τ_U given by

$$\tau_U = \inf\left\{t > 0 \mid X_t \in U\right\}. \quad (25)$$

Since the inequality (20) holds on $\mathbb{R} \setminus U$, according to (20) in the condition (CO2) and Theorem 3.9 of Khasminskii (2012), we obtain that

$$\mathrm{E}\left\{\tau_U\right\} < \infty. \quad (26)$$

This implies that

$$\mathrm{P}\left\{\tau_U < \infty\right\} = 1. \quad (27)$$

Therefore, we can conclude that the state X_t reaches U in finite time with probability one. Because $U \subset I$ holds and the solution X_t is continuous in t, we obtain that when the initial value x is given so that $x \notin I$, $\tau_U \geq \tau_I$ holds almost surely. Therefore, we can conclude the property (22).

Lastly, we show the property (23). Because τ_I is given by $\tau_I = \inf\left\{t > 0 \mid X_t \in I\right\}$ and the property (21) in the condition (CO3) holds, for $t \geq \tau_I$, by using Theorem 3, the property (23) immediately follows.

When $x \in I$, the argument (24) immediately follows from Theorem 3. This completes the proof.

5. CONDITIONS OF BOUNDEDNESS OF N-DIMENSIONAL SYSTEMS

This section extends the result in Section 4 for general n-dimensional systems. Due to the limited space, we only discuss the boundedness and only consider systems with one-dimensional Wiener processes.

This section provides a condition of the boundedness in the almost sure sense for the system (4). The result is a modification of the non-explosion condition by Khasminskii, which can be found in (Rogers and Williams, 2000, Chapter V, Section 7).

To provide the condition, we introduce the following notations:

$$\begin{aligned} \mu(r) &= \sup_{\|x\|=r} \left\{\sigma(x)^T\sigma(x) + 2x^T f(x)\right\}, \\ \theta(r) &= \sup_{\|x\|=r} \left\{2\|x^T\sigma(x)\|\right\}, \end{aligned} \tag{28}$$

where $f(x)$ and $\sigma(x)$ is given in (4). Moreover, we introduce the functions

$$\begin{aligned} s'(r) &= \exp\left(-2\int_c^r \frac{\mu(r)}{\theta(r)^2}dr\right), s(c) = 0, \\ m'(r) &= \frac{1}{\theta(r)^2 s'(r)}. \end{aligned} \tag{29}$$

where $c > 0$ is the base point of the integration.

The following theorem provides the condition of the boundedness.

Theorem 5. Consider the system (4). With the functions $\mu(r)$ and $\theta(r)$ of (28), assume that there exists $\bar{r} > 0$ such that for $c < \bar{r}$,

$$\int_c^{\bar{r}} s'(x)dx \int_c^x m'(y)dy = \infty, \tag{30}$$

holds where $s'(r)$ and $m'(r)$ are given in (29), and assume that $\|X_0\| < \bar{r}^{1/2}$. Then,

$$\mathrm{P}\left\{\sup_t \|X_t\| < \bar{r}^{1/2}\right\} = 1 \tag{31}$$

holds.

Proof. We provide an outline of the proof. Let $R_t = \|X_t\|^2$. The proof is given by showing the existence of a function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $u(r)$ is nonnegative and increasing with respect to $r \in \mathbb{R}_{\geq 0}$ and $u(r) \to \infty$ as $r \to \bar{r}$, and by showing that $e^{-t}u(R_t)$ is a nonnegative supermartingale, we will obtain (31).

We first consider to construct a function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is nonnegative and increasing and solves

$$u(r) = \mu(r)u'(r) + \frac{1}{2}\theta(r)^2 u''(r), \quad u(c) = 1, \tag{32}$$

where $r \in \mathbb{R}_{\geq 0}$. And $c \in \mathbb{R}_{\geq 0}$ is a sufficiently small parameter. As shown in the proof of Theorem 52.1 of Rogers and Williams (2000), there exists such a nonnegative and increasing function $u(r)$. Moreover, the function $u(r)$ can be constructed to be C^2 and convex. In addition, it can be shown that $u(r) \to \infty$ as $r \to \bar{r}$ when (30) holds.

Then, by using the Itô formula for R_t and the system (4), we obtain that

$$dR_t = \left[2X_t^T f(X_t) + \sigma(X_t)^T\sigma(X_t)\right]dt + 2X_t^T\sigma(X_t)dW_t. \tag{33}$$

By denoting the infinitesimal generator of (33) as $\bar{\mathcal{L}}$, we obtain that for the function u obtained above,

$$\begin{aligned} \bar{\mathcal{L}}(u(R_t)e^{-t}) = e^{-t}u'(R_t)\left[2X_t^T f(X_t) + \sigma(X_t)^T\sigma(X_t)\right] \\ + 2e^{-t}u''(R_t)\left(X_t^T\sigma(X_t)\right)^2 - e^{-t}u(R_t) \end{aligned} \tag{34}$$

Because the function $u(r)$ is increasing and convex in r, the definitions of $\mu(r)$ and $\nu(r)$ give the inequality

$$\begin{aligned} &\bar{\mathcal{L}}(u(R_t)e^{-t}) \\ &\leq e^{-t}\left[\mu(R_t)u'(R_t) + \frac{1}{2}\theta(R_t)^2u''(R_t) - u(R_t)\right]. \end{aligned} \tag{35}$$

Then, because $u(r)$ solves (32), $\bar{\mathcal{L}}\left(u(R_t)e^{-t}\right) \leq 0$ holds. By introducing

$$\tau_n = \inf\left\{t; \|X_t\|^2 > \bar{r} - 1/n\right\}, \tag{36}$$

we can conclude that $u(R_{t\wedge\tau_n})e^{-t\wedge\tau_n}$ is a supermartingale process. Therefore, according to the property of the supermartingale and the definition of τ_n, it holds that

$$u(R_0) \geq \mathrm{E}\left[u(R_{\tau_n})e^{-\tau_n}\right] = u(\bar{r} - 1/n)\mathrm{E}\left[e^{-\tau_n}\right]. \tag{37}$$

This implies that

$$\mathrm{E}\left[e^{-\tau_n}\right] \leq \frac{u(R_0)}{u(\bar{r} - 1/n)}. \tag{38}$$

Then, according to Fatou's lemma, we can show that

$$\mathrm{E}\left[e^{-\tau_n}\right] \leq \frac{u(R_0)}{u(\bar{r} - 1/n)} \to 0 \text{ as } n \to \infty.$$

because $u(r) \to \infty$ as $r \to \bar{r}$ under the condition (30). This implies that (31) holds.

Remark 1. The n-dimensional cases employ an auxiliary function $R_t = \|X_t\|^2$ to investigate the boundedness. This causes some differences from the results for one-dimensional cases in Section 4.

6. NUMERICAL EXAMPLE

We show the continuation of the example given by (3) in Section 2. We apply the results of preceding sections to this example to validate the presented results.

We first apply Theorem 3 to investigate the interval $(0, 1)$. To this end, we show that $\nu(0+) = \infty$ and $\nu(1-) = \infty$ in the following. Straightforward calculations show that

$$\nu(0+) = \int_0^{1/2}\left\{\int_\xi^{1/2} x(1-x)dx\right\}\frac{d\xi}{\xi^2(1-\xi)^2}, \tag{39}$$

and we obtain $\nu(0+) = \infty$. Similarly, we can obtain that $\nu(1-) = \infty$. Moreover, based on Karlin and Taylor (1981), straightforward calculations show that

$$\mu(0+) = \int_0^{1/2}\left\{\int_\xi^{1/2}\frac{dx}{x^2(1-x)^2}\right\}(\xi(1-\xi))\,d\xi, \tag{40}$$

and this indicates $\mu(0+) < \infty$. Similar calculations give $\mu(1-) < \infty$. Therefore, according to Theorem 3, we can conclude that

$$\mathrm{P}\left\{X_t \in (0, 1) \text{ for } t \geq 0\right\} = 1, \tag{41}$$

when $X_0 = x \in (0, 1)$.

Then, we apply Theorem 4 to the system (3). We already have shown that $\nu(0+) = \infty$, $\nu(1-) = \infty$, $\mu(0+) < \infty$, and

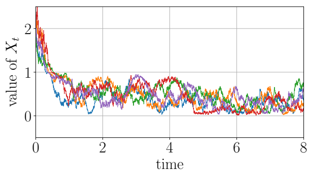

Fig. 2. Sample Paths of X_t of (3) with $x = 2$

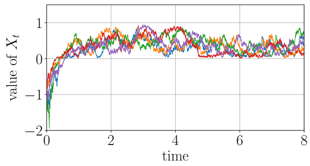

Fig. 3. Sample Paths of X_t of (3) with $x = -1$

$\mu(1-) < \infty$. Moreover, to apply Theorem 4, we consider a function $V : \mathbb{R} \to \mathbb{R}$ given by

$$V(x) = \left(x - \frac{1}{2}\right)^2. \tag{42}$$

With the infinitesimal generator $\mathcal{L}$ of the system (3), we can obtain that

$$\mathcal{L}V(x) = 2\left(x - \frac{1}{2}\right)(1 - 2x) + |x(1 - x)|. \tag{43}$$

We can find the set U in Theorem 4 as $(0.1, 0.9)$ and $\mathcal{L}V(x) < -k$ with $k = 1/2^4$ for $x \in \mathbb{R} \setminus U$. Therefore, we can conclude the properties given in Theorem 4 for the system (3). Figs. 2 and 3 show time responses of the state of the system (3) with the initial values $x = 2$ and $x = -1$, respectively and five sample paths are shown in each figure. These sample paths are obtained by using the Euler–Maruyama method. These figures illustrate the state of the system (3) converges to the set $I = (0, 1)$ even when the initial value x is not in the set I. This implies the practical asymptotic stability.

7. CONCLUSIONS

This paper investigated the boundedness and the practical asymptotic stability for stochastic systems described by stochastic differential equations. As seen in the example of Section 2, in general, stochastic systems do not exhibit either the boundedness or the convergence to the bounded region. This study reported a sufficient condition of the boundedness and practical asymptotic stability for a class of stochastic systems. The results are based on the properties of boundaries of one-dimensional stochastic processes. Although this study provided only a condition for the boundedness of n-dimensional stochastic systems, we can also extend a condition of the practical asymptotic stability to n-dimensional cases by using Lyapunov-like functions, which will be included in future works. Future works also include applications to the control problems such as guaranteeing the almost sure boundedness.

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