Newton's method: sufficient conditions for practical and input-to-state stability ⋆

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Abstract: Newton's method is a classical iterative algorithm for the numerical computation of isolated roots of algebraic equations and stationary points of functions. While its application is ubiquitous in a plethora of fields, questions concerning its robust stability to uncertainties in problem data and numerical accuracy often arise in practice. This paper seeks to provide sufficient conditions for practical stability, input-to-state-stability (ISS), integral ISS (iISS) and incremental ISS (δISS) of Newton's method in the presence of such uncertainties, and provide illustrative examples of their application.

Keywords: Numerical methods, Newton's method, robust stability, input-to-state stability (ISS), integral ISS (iISS), incremental ISS (δISS).

1. INTRODUCTION

Newton's method is a classical iterative algorithm for solving algebraic equations, see for example Nocedal and Wright (2006). It finds widespread application across a broad range of disciplines, and is of particular utility where numerical optimization is required. A typical problem concerns the numerical computation of an isolated stationary point of some sufficiently regular function $f : \mathbb{R}^n \to \mathbb{R}$, i.e. finding an $x^* \in \mathcal{X} \subset \mathbb{R}^n$ (possibly non-unique) such that

$$0 = \nabla f(x^*), \qquad (1)$$

in which $\nabla f : \mathcal{X} \to \mathbb{R}^n$ is the gradient of f.

An iterative algorithm, such as Newton's method, is a discrete time dynamical system. Using a suitably defined state transition mapping $g : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$, its k^{th} iteration step is given by

$$x_{k+1} = g(x_k, u_k), \quad k = 0, 1, 2, \ldots, \qquad (2)$$

in which $x_k \in \mathbb{R}^n$ describes the algorithm state, as evolved from some initial state $x_0 \in \mathbb{R}^n$, in the presence of an exogenous input $u_k \in \mathcal{U}$ to the algorithm, drawn from a bounded exogenous input sequence. Mapping g describes the algorithm, while a generated trajectory $\{x_k\}_{k=0,1,2,\cdots}$ describes an execution of the algorithm in the presence of an exogenous input sequence $\{u_k\}_{k=0,1,2,\cdots}$, given an initial state x_0.

A classical Newton's method for solving (1) is an iterative algorithm described by the discrete-time dynamical system (2), evolving on $\mathbb{R}^n$, with $g : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ defined by

$$g(x, u) \doteq x - H(x)^{-1} \nabla f(x), \qquad (3)$$

for all $x \in \mathbb{R}^n$, $u \in \mathcal{U}$, with $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ and $H \doteq D \nabla f : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ denoting the gradient and Hessian of f, and D the (Fréchet) derivative (see Long (2009)) (noting that the input space $\mathcal{U}$ is unused for (3)). Using representation (2), (3), Newton's method can be studied within the framework of systems theory, and in particular the well-developed tools of robust stability theory may be applied. For example, (2), (3) may be interpreted as a feedback interconnection of a linear dynamical system and a static nonlinear map, as per Figure 1, and its behaviour analyzed using control systems analysis and design tools.

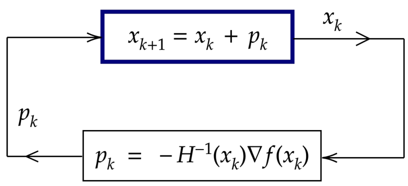

Fig. 1. Newton's method (2), (3) as a feedback system.

In the literature, there are many results concerning the stability of Newton's method (2), (3), and related iterative algorithms, with respect to an exact stationary point $x^* \in \mathbb{R}^n$ in the absence of uncertainty; see for example Luenberger and Ye (2007); Kelley (1999); Bertsekas (2015). In implementation, Newton's method is subject to uncertainties in problem data, numerical approximations, computation errors, etc. Collectively, these uncertainties can be regarded as *disturbances* that impact algorithm behaviour. Such disturbances can substantially degrade the qualitative convergence properties of Newton's method, compared with the disturbance-free case, via changes to the domain of attraction, convergence rate, limit set, etc. Consequently, analysis of the robust stability of Newton's method is a key theoretical consideration that should underpin its practical implementation. Moreover, in view of the nonlinear feedback interconnection representation of Figure 1, and the plethora of robust stability concepts

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and tools available for nonlinear dynamical systems, it is reasonable to frame this analysis in a nonlinear dynamical systems context. Existing analysis in this direction includes the development of Lipschitz-like sufficient conditions on the step-size for convergence of Newton's method applied to nonlinear equations of a single variable, see for example Kashima and Yamamoto (2007), etc.

The focus of this paper is the development of sufficient conditions for ensuring *practical stability*, *input-to-state stability (ISS)*, and related robust stability properties, for Newton's method; see for example Sontag (1989), Jiang and Wang (2001), Tran et al. (2017), Tran et al. (2018), Guiver and Logemann (2020), and Hasan et al. (2013), Coughlan et al. (2007), Sarkens and Logemann (2016) for recent related results. Section 2 provides a practical stability result that takes into account disturbances in the iteration step of Newton's method (2), (3), and guarantees convergence of its state x_k to a ball centred on the stationary point x^*. Results for guaranteeing input-to-state stability (ISS), integral ISS (iISS), and incremental ISS (δISS) are provided for Newton's method (2), (3) in Section 3, for similarly restricted classes of functions f.

In terms of notation, the sets of naturals, integers, and real numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$. The sets of nonnegative integers and non-negative reals are likewise denoted by $\mathbb{Z}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$. Similarly, n-dimensional Euclidean space and the space of $n \times m$ matrices on $\mathbb{R}$ are denoted respectively by $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, for $n, m \in \mathbb{N}$. The associated Euclidean and induced norms are both denoted by $\|\cdot\|$. An open ball of radius $r \in \mathbb{R}_{>0}$ centred at $x \in \mathbb{R}^n$ is denoted by $\mathcal{B}(x; r)$, and the distance of $x \in \mathbb{R}^n$ from a set $X \subset \mathbb{R}^n$ is $d(x, X) \doteq \inf_{y\in X} \|y-x\|$. The space of bounded sequences in $\mathbb{R}^m$ is denoted by ℓ^m_∞, $n, m \in \mathbb{N}$, while the corresponding norm of $u \in \ell^m_\infty$ is denoted by $\|u\|_\infty$. The space of bounded linear operators between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ is denoted by $\mathscr{L}(\mathcal{X};\mathcal{Y})$. A continuous function $\gamma\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is positive definite if $\gamma(0) = 0$ and $\gamma(r) > 0$ for all $r > 0$. The class of all positive functions is denoted by $\mathcal{P}$. If, in addition, γ is strictly increasing then it is said to be of class $\mathcal{K}$. If $\gamma \in \mathcal{K}$ and $\lim_{r\to\infty} \gamma(r) = \infty$, then γ is said to be of class $\mathcal{K}_\infty$. Meanwhile, a strictly decreasing continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\lim_{r\to\infty} \gamma(r) = 0$ is said to be of class $\mathcal{L}$, and a continuous function $\beta\colon [0, a) \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $a \in \mathbb{R}_{>0}$, is said to belong to class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed t and $\beta(r, \cdot) \in \mathcal{L}$ for each fixed r. For the discrete-time dynamical system (2), a trajectory evolving in $\mathbb{R}^n$ from initial condition $x_0 \in \mathbb{R}^n$, and driven by input $u \in \ell^m_\infty$, is denoted at time step $k \in \mathbb{Z}_{\geq 0}$ by $x(k, x_0, u)$.

2. PRACTICAL STABILITY OF NEWTON'S METHOD

2.1 Quadratic convergence and asymptotic stability

Under the reasonable conditions, see for example (Nocedal and Wright, 2006, Theorem 3.5), Newton's method (2), (3) applied to the stationary point problem of (1) converges to an $x^* \in \mathbb{R}^n$, i.e. $\lim_{k\to\infty} x_k = x^*$ (and x^* may depend on the choice of $x_0 \in \mathbb{R}^n$). Here, it is convenient to assume that $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ in (1), (2) satisfy

$$
\begin{aligned}
&\exists \text{ simple, connected, open } \mathcal{X} \subset \mathbb{R}^n \text{ and } c \in [0,1), \text{ s.t.}\\
&\star\ f \in C^3(\mathcal{X};\mathbb{R});\\
&\star\ \exists\, x^* \in \mathcal{X} \text{ s.t. (1) holds; and} \qquad (4)\\
&\star\ x \mapsto H(x) \doteq D\nabla f(x) \in \mathbb{R}^{n\times n} \text{ satisfies}\\
&\quad \sup_{x\in\mathcal{X}} \max(\|H(x)\|,\, \|H(x)^{-1}\|,\, \|DH(x)\|_{\mathscr{L}(\mathbb{R}^n;\mathbb{R}^n\times\mathbb{R}^n)}) \leq c.
\end{aligned}
$$

In view of (4), it is useful to define $\mathcal{X}_0 \subset \mathcal{X}$ by

$$
\begin{aligned}
\mathcal{X}_0 &\doteq \mathcal{B}(x^*; \delta^H_{\mathcal{X}}), \qquad (5)\\
\delta^H_{\mathcal{X}} &\doteq \sup\{r \in (0, \tfrac{1}{c^2}) \,|\, \mathcal{B}(x^*; r) \subset \mathcal{X}\}.
\end{aligned}
$$

The following statements are then standard.

Lemma 1. Let f, $x^* \in \mathcal{X}$, $c \in [0, 1)$ satisfy (4). Then, $\|g(x, 0) - x^*\| \leq c^2 \|x - x^*\|^2$ for all $x \in \mathcal{X}$, in which g is as per (3).

Proof. Fix $x^* \in \mathcal{X}$ as per (1), (4), and $x \in \mathcal{X}$. By the asserted regularity of f, note further by (4) that $H(x)$, $H(x)^{-1} \in \mathbb{R}^n\times\mathbb{R}^n$ and $R(x, x^*) \in \mathscr{L}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$, where $H(x) \doteq D\nabla f(x)$, $R(x, x^*) \doteq \int_0^1 (1-t)\, DH(x+t(x^*-x))\, dt$. Applying Taylor's theorem yields $0 = \nabla f(x^*) = \nabla f(x) + H(x)\,(x^*-x) + R(x, x^*)\,(x^*-x)\,(x^*-x)$, so that by (4), $x^* - g(x, 0) = -H(x)^{-1} R(x, x^*)\,(x^*-x)\,(x^*-x)$. Taking the norm, applying the triangle inequality and (4), $\|x^*-g(x, 0)\| \leq \|H(x)^{-1}\|\,\|R(x, x^*)\|\,\|x^*-x\|^2 \leq c^2\,\|x^*-x\|^2$. As $x \in \mathcal{X}$ is arbitrary, the assertion follows. □

Theorem 2. Given $x^* \in \mathcal{X}$ and $c \in [0, 1)$ satisfying (1), (4), and any $x_0 \in \mathcal{X}_0 \subset \mathcal{X}$, c.f. (5), Newton's method (2), (3) is quadratically convergent to x^*, i.e.

$$
\|x_{k+1} - x^*\| \leq c^2\,\|x_k - x^*\|^2, \qquad (6)
$$

for all $k \in \mathbb{N}$, with $\lim_{k\to\infty} x_k = x^*$.

Proof. Fix $x^* \in \mathcal{X}$ and $c \in [0, 1]$ as per (1), (4), and note that $x^* \in \mathcal{X}_0$ by (5). Fix any $x_0 \in \mathcal{X}_0$, $k \in \mathbb{N}$. Let $x_k \doteq x(k, x_0, 0)$ be as per (2), (3), and suppose that $x_k \in \mathcal{X}_0$. Let $x_{k+1} \doteq g(x_k, 0)$, and observe by Lemma 1 and (5) that $\|g(x_k, 0) - x^*\| \leq c^2\,\|x_k - x^*\|^2 < c^2(\delta^H_{\mathcal{X}})^2 < \delta^H_{\mathcal{X}}$, i.e. $x_{k+1} \in \mathcal{X}_0$. Hence, $x_k \in \mathcal{X}_0 \subset \mathcal{X}$ for all $k \in \mathbb{Z}_{\geq 0}$, so that (6) follows by Lemma 1, and the asserted limit follows. □

2.2 Additive uncertainties and practical stability

The convergence property provided by Theorem 2 is not guaranteed in the presence of bounded round-off and other additive approximation errors. Indeed, the numerical approximation x_k may instead converge to some neighbourhood of x^*, or it may diverge. Sufficient conditions that guarantee the former case, referred to here as *practical stability*, along with quantifiable bounds on the size of the ω-limit set obtained, are thus of interest.

Two sources of additive uncertainty are considered, due to the numerical evaluation of $H(x)^{-1}$ and $\nabla f(x)$. These uncertaintiess are assumed uniformly bounded a priori, via the pair $\delta \doteq (\delta^H, \delta^f) \in \mathbb{R}^2_{\geq 0}$ for $H(\cdot)^{-1}$ and $\nabla f(\cdot)$. The corresponding generalization of Newton's method (2), (3) is defined via $g : \mathcal{X} \times \mathcal{U}^\delta \to \mathcal{X}$ by

$$
g(x, u) \doteq x - \left(H(x)^{-1} + u^H\right)\left(\nabla f(x) + u^f\right), \qquad (7)
$$

for all $x \in \mathcal{X} \doteq \mathbb{R}^n$, $u \in \mathcal{U}^\delta$, in which

$$
\mathcal{U}^\delta \doteq \left\{ u \doteq (u^H,\, u^f) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \;\middle|\; \begin{array}{l} \|u^H\| \leq \delta^H \\ \|u^f\| \leq \delta^f \end{array} \right\}, \qquad (8)
$$

A space of bounded disturbance (input) sequences is defined via (8) by $\mathscr{U}^\delta \doteq \{u : \mathbb{Z}_{\geq 0} \to \mathcal{U}^\delta\} \subset \ell^m_\infty \times \ell^m_\infty$.

A feedback representation corresponding to Figure 1 for the perturbed Newton's method (2), (7) is illustrated in Figure 2, in which $u_k \doteq (u_k^H, u_k^f) \in \mathcal{U}^\delta$ for all $k \in \mathbb{Z}_{\geq 0}$.

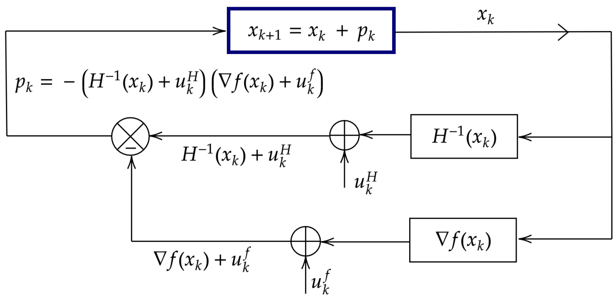

Fig. 2. Perturbed Newton's method (2), (7).

In view of (4), and using (5), it is convenient to further define

$$\mathcal{X}_1 \doteq \mathcal{B}(x^*; \min(\delta_\mathcal{X}^H,\ \delta_\mathcal{X}^f)) \subset \mathcal{X}_0, \tag{9}$$

$$\delta_\mathcal{X}^f \doteq \sup\left\{ \delta_1 > 0 \;\middle|\; \begin{array}{c} \|\nabla f(x)\| \leq 2\,\|H(x^*)\|\,\|x - x^*\| \\ \forall\, x \in \mathcal{B}(x^*;\delta_1) \end{array} \right\},$$

Practical stability for a perturbed Newton's method (2), (7) may be asserted via the following lemma and theorem.

Lemma 3. Given g as per (7), $x^* \in \mathcal{X}$, $c \in [0,1)$ as per (4), and any $\delta \doteq (\delta^H, \delta^f) \in \mathbb{R}_{\geq 0}^2$,

$$\|g(x,u) - x^*\| \leq c^2\,\|x - x^*\|^2 + 2\,\delta^H\,c\,\|x - x^*\| + (c + \delta^H)\,\delta^f, \tag{10}$$

for all $x \in \mathcal{X}_1$, $u \in \mathcal{U}^\delta$.

Proof. Fix $x^* \in \mathcal{X}$, $c \in [0,1)$ as per (1), (4), and $x \in \mathcal{X}_1$. Note by (9) that $x^* \in \mathcal{X}_1$. Fix any $\delta \doteq (\delta^H, \delta^f) \in \mathbb{R}_{\geq 0}^2$, $u \in \mathcal{U}^\delta$. By inspection of (2), (7), observe that

$$\begin{aligned} g(x,u) - x^* &= x - H(x)^{-1}\,\nabla f(x) - x^* \\ &\quad - H(x)^{-1}\,u^f - u^H\,\nabla f(x) - u^H\,u^f \\ &= g(x,0) - x^* - H(x)^{-1}\,u^f - u^H\,\nabla f(x) - u^H\,u^f, \end{aligned}$$

so that by the triangle inequality and Lemma 1,

$$\begin{aligned} \|g(x,u) - x^*\| &\leq \|g(x,0) - x^*\| + \|H(x)^{-1}\|\,\delta^f \\ &\quad + \|\nabla f(x)\|\,\delta^H + \delta^H\,\delta^f \\ &\leq c^2\,\|x - x^*\|^2 + 2\,\delta^H\,c\,\|x - x^*\| + (c + \delta^H)\,\delta^f, \end{aligned}$$

as required by (10). □

Theorem 4. Let $x^* \in \mathcal{X}$, $c \in (0,1)$ satisfy (1), (4), and $\delta \doteq (\delta^H, \delta^f) \in \mathbb{R}_{\geq 0}^2$ satisfy

$$\delta^H < \tfrac{1}{2\,c},\ \ \delta^f < \tfrac{(1 - 2\,\delta^H\,c)^2}{4\,c^2\,(c + \delta^H)},\ \ \varepsilon_-^c(\delta) \leq \min(\delta_\mathcal{X}^H, \delta_\mathcal{X}^f), \tag{11}$$

with $\varepsilon_-^c(\delta) \doteq \inf\{\varepsilon > 0 \,|\, p_\delta^c(\varepsilon) < 0\}$, $\varepsilon_+^c(\delta) \doteq \sup\{\varepsilon > 0 \,|\, p_\delta^c(\varepsilon) < 0\}$, and $p_\delta^c(\varepsilon) \doteq c^2\,\varepsilon^2 + (2\,\delta^H\,c - 1)\,\varepsilon + (c + \delta^H)\,\delta^f$ for all $\varepsilon \in \mathbb{R}$. Then, for any $x_0 \in \mathcal{X}_1 \cap \mathcal{B}(x^*; \varepsilon_+^c(\delta))$, $u \in \mathscr{U}^\delta$, the perturbed Newton's method (2), (7) converges to $\mathcal{X}_1 \cap \mathcal{B}(x^*; \varepsilon_-^c(\delta))$, i.e. $\lim_{k\to\infty} d(x_k, \mathcal{X}_1 \cap \mathcal{B}(x^*; \varepsilon_-^c(\delta))) = 0$.

Proof. Fix $x^* \in \mathcal{X}$ as per (1), (4), and note that $x^* \in \mathcal{X}_1$ by (9). Fix any $\delta \doteq (\delta^H, \delta^f) \in \mathbb{R}_{\geq 0}^2$ such that the first two inequalities in (11) hold. Note that

$$\varepsilon_-^c(\delta) = \tfrac{1 - 2\,\delta^H\,c - \sqrt{(1 - 2\,\delta^H\,c)^2 - 4\,c^2\,(c + \delta^H)\,\delta^f}}{2\,c^2} > 0,\ \ \delta > 0,$$

$\varepsilon_-^c(0) = 0$, and $\varepsilon_-^c : \mathbb{R}_{\geq 0}^2 \to \mathbb{R}_{\geq 0}$ is continuous by inspection. Hence, it is always possible to choose δ such that all three inequalities in (11) hold. Fix any $u \in \mathcal{U}^\delta$, $x_0 \in \mathcal{X}_1 \cap \mathcal{B}(x^*; \varepsilon_+^c(\delta))$, $k \in \mathbb{N}$. Define $x_k \doteq x(k, x_0, u)$ as per (2), (7), and suppose that $x_k \in \mathcal{X}_1 \cap \mathcal{B}(x^*; \varepsilon_+^c(\delta))$. Let $v(y) \doteq \|y - x^*\|$ for all $y \in \mathbb{R}^n$, $x_{k+1} \doteq g(x_k, u_k)$, and observe by Lemma 3 that $v(x_{k+1}) = \|g(x_k, u_k) - x^*\| \leq c^2\,v(x_k)^2 + 2\,\delta^H\,c\,v(x_k) + (c + \delta^H)\,\delta^f$, i.e. $v(x_{k+1}) - v(x_k) \leq p_\delta^c(v(x_k))$. Hence, $x_{k+1} \in \mathcal{X}_1 \cap \mathcal{B}(x^*; \varepsilon_+^c(\delta))$, and $k \mapsto v(x_k)$ decreases for all k s.t. $x_k \notin \mathcal{X}_1 \cap \mathcal{B}(x^*; \varepsilon_-^c(\delta))$, yielding the stated convergence. □

Remark 5. Theorem 4 may be specialized to three cases, corresponding to knowledge of the exact inverse of the Hessian, or the exact gradient, or both. With $\delta \doteq (\delta^H, \delta^f) \in \mathbb{R}_{\geq 0}^2$, these cases correspond to selecting $\delta^H = 0$, $\delta^f = 0$, or $\delta^H = 0 = \delta^f$, respectively.

$\delta^H = 0$: Let $\Delta^c \doteq 2\,c^2\,\min(\delta_\mathcal{X}^H,\ \delta_\mathcal{X}^f)$, with $\delta_\mathcal{X}^H$, $\delta_\mathcal{X}^f$ as per (5), (9). Note that $2\,c^2\,\varepsilon_\pm^c(\delta) = 1 \pm \sqrt{1 - 4\,c^3\,\delta^f}$, and the third inequality in (11) is $1 - \sqrt{1 - 4\,c^3\,\delta^f} \leq \Delta^c$. Hence, selecting δ_f such that

$$\delta^f < \begin{cases} \frac{1}{4\,c^3}, & \text{if } \Delta^c \geq 1, \\ \frac{1 - (1 - \Delta^c)^2}{4\,c^3}, & \text{if } \Delta^c < 1, \end{cases}$$

implies that the conditions (11) required for Theorem 4 to hold are satisfied. Hence, given any $x_0 \in \mathcal{B}(x^*; \varepsilon_+^c(\delta))$, Theorem 4 implies that $\lim_{k\to\infty} d(x_k, \mathcal{B}(x^*; \varepsilon_-^c(\delta))) = 0$.

$\delta^f = 0$: With $\delta^H < \frac{1}{2\,c}$, note that $\varepsilon_-^c(\delta) = 0$, $\varepsilon_+^c(\delta) = \frac{1 - 2\,\delta^H\,c}{c^2}$, and that the conditions (11) of Theorem 4 are satisfied. Hence, given any $x_0 \in \mathcal{B}(x^*; \varepsilon_+^c(\delta))$, Theorem 4 implies that $\lim_{k\to\infty} d(x_k, \{x^*\}) = 0$, i.e. convergence to x^* is guaranteed.

$\delta^H = 0 = \delta^f$: $\varepsilon_-^c(\delta) = 0$, $\varepsilon_+^c(\delta) = \frac{1}{c^2}$, and the conditions of (11) of Theorem 4 are automatically satisfied. Hence, given any $x_0 \in \mathcal{B}(x^*; \frac{1}{c^2})$, Theorem 4 implies that $\lim_{k\to\infty} d(x_k, \{x^*\}) = 0$. This recovers the statement of Theorem 2 for the unperturbed Newton's method (2), (3).

Example 6. The perturbed Newton's method (2), (7) is applied to the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \doteq 100\,(y - x^2)^2 + (x - 1)^2,$$

for all $(x,y) \in \mathbb{R}^2$. Figure 3 illustrates the convergence of an ensuing iteration to a neighbourhood of its minimum at $(x,y) = (1,1)$, for the case where $\delta^H \doteq 0$ and $\delta^f \doteq 10^{-7}$. An a priori convergence bound is illustrated by the red circle of radius 5×10^{-7}.

2.3 A local Hessian approximation and practical stability

Where a invertible local approximation $x \mapsto \widetilde{H}(x) \in \mathbb{R}^{n\times n}$ for the Hessian of f is available, and a uniformly bounded iteration uncertainty is present, an alternative to the generalization (2), (3) may be more appropriate. In particular, with an uncertainty bound $\delta^v \in \mathbb{R}_{\geq 0}$, the corresponding generalization of Newton's method (2), (3), is defined via $g : \mathcal{X} \times \mathcal{V}^\delta \to \mathcal{X}$ by

$$g(x,v) \doteq x - \widetilde{H}(x)^{-1}\,\nabla f(x) + v, \tag{12}$$

for all $x \in \mathcal{X} \doteq \mathbb{R}^n$, $v \in \mathcal{V}^\delta$, in which

$$\mathcal{V}^\delta \doteq \{v \in \mathbb{R}^n \,|\, \|v\| \leq \delta\}, \tag{13}$$

and the corresponding space of bounded disturbance sequences is $\mathscr{V}^\delta \doteq \{v : \mathbb{Z}_{\geq 0} \to \mathcal{V}^\delta\} \subset \ell_\infty^m$. Given $\mu \in \mathbb{R}_{\geq 0}$, $\mu < 1$, and $\omega \in \mathbb{R}_{>0}$ fixed, it is convenient to define

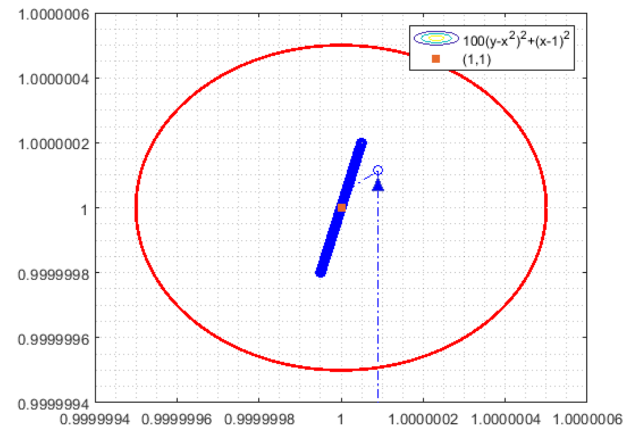

Fig. 3. Practical stability for Newton's Method.

$$\mathcal{X}_2 \doteq \mathcal{B}(x^*; \tilde{\delta}_{\mathcal{X}}^H) \subset \mathbb{R}^n, \tag{14}$$

$$\tilde{\delta}_{\mathcal{X}}^H \doteq \sup \left\{ \tilde{\delta} > 0 \;\middle|\; \begin{array}{c} \|\widetilde{H}(x)^{-1} (H(y) - H(x))\| \leq \omega \|y - x\| \\ \|\widetilde{H}(x)^{-1} (\widetilde{H}(x) - H(x))\| \leq \mu \\ \forall\, x, y \in \mathcal{B}(x^*; \tilde{\delta}) \end{array} \right\}$$

in which $H(\cdot)$, $\widetilde{H}(\cdot)$ are as per (12).

Theorem 7. Let g be as per as (12), $x^* \in \mathcal{X}$ satisfy (4), and $\delta \in \mathbb{R}_{\geq 0}$ such that

$$\delta < \tfrac{(1-\mu)^2}{2\,\omega}\,. \tag{15}$$

Then, for any $x_0 \in \mathcal{X}_2 \cap \mathcal{B}(x^*; \varepsilon_+^{\mu,\omega}(\delta))$, $v \in \mathscr{V}^\delta$, with

$$\begin{aligned} \varepsilon_-^{\mu,\omega}(\delta) &\doteq \inf \{ \varepsilon > 0 \,|\, p_\delta^{\mu,\omega}(\varepsilon) < 0 \}, \\ \varepsilon_+^{\mu,\omega}(\delta) &\doteq \sup \{ \varepsilon > 0 \,|\, p_\delta^{\mu,\omega}(\varepsilon) < 0 \}, \\ p_\delta^{\mu,\omega}(\varepsilon) &\doteq \tfrac{1}{2}\,\omega\,\varepsilon^2 + (\mu - 1)\,\varepsilon + \delta, \end{aligned} \tag{16}$$

the perturbed approximate Newton's method (2), (12) convergences to $\mathcal{X}_2 \cap \mathcal{B}(x^*; \varepsilon_-^{\mu,\omega}(\delta))$, i.e. $\lim_{k\to\infty} d(x_k, \mathcal{X}_2 \cap \mathcal{B}(x^*; \varepsilon_-^{\mu,\omega}(\delta))) = 0$.

Proof. Fix $x^* \in \mathcal{X}$ as per (1), (4), and note that $x^* \in \mathcal{X}_2$ by (14). Fix any $\delta \in \mathbb{R}_{\geq 0}$ such (15) holds, $x_0 \in \mathcal{X}_2 \cap \mathcal{B}(x^*; \frac{1}{\omega})$, and $v \in \mathscr{V}^\delta$. Recalling (12),

$$\begin{aligned} g(x, v) - x^* &= x - \widetilde{H}(x)^{-1}\,\nabla f(x) + v - x^* \\ &= \widetilde{H}(x)^{-1}\,\widetilde{H}(x)\,(x - x^*) - \widetilde{H}(x)^{-1}\,\nabla f(x) + v \\ &= \widetilde{H}(x)^{-1}\,(\widetilde{H}(x) - H(x))\,(x - x^*) + v \\ &\quad - \widetilde{H}(x)^{-1} \left(\textstyle\int_0^1 H(x^* + t(x - x^*)) - H(x)\,dt \right) (x - x^*). \end{aligned}$$

By the triangle inequality,

$$\begin{aligned} &\|g(x, v) - x^*\| \leq \|\widetilde{H}(x)^{-1}\,(\widetilde{H}(x) - H(x))\|\,\|x - x^*\| + \|v\| \\ &\quad + \textstyle\int_0^1 \|\widetilde{H}(x)^{-1}\,(H(x^* + t(x - x^*)) - H(x))\|\,dt\,\|x - x^*\| \\ &\leq \mu\,\|x - x^*\| + \delta + \textstyle\int_0^1 \omega\,(1 - t)\,\|x - x^*\|\,dt \|x - x^*\| \\ &= \tfrac{1}{2}\,\omega\,\|x - x^*\|^2 + \mu\,\|x - x^*\| + \delta\,. \end{aligned} \tag{17}$$

Let $x_k \doteq x(k, x_0, v)$ as per (2), (12), and suppose that $x_k \in \mathcal{X}_2 \cap \mathcal{B}(x^*; \varepsilon_+^{\mu,\omega}(\delta))$. Let $v(y) \doteq \|y - x^*\|$ for all $y \in \mathbb{R}^n$, $x_{k+1} \doteq g(x_k, v_k)$, and observe by (17) that

$$\begin{aligned} v(x_{k+1}) = \|g(x_k, v_k) - x^*\| &\leq \tfrac{1}{2}\,\omega\,v(x_k)^2 + \mu\,v(x_k) + \delta \\ &\implies v(x_{k+1}) - v(x_k) \leq p^{\mu,\omega}(v(x_k)), \end{aligned}$$

with $p^{\mu,\omega}(\cdot)$ as per (16). Hence, $x_{k+1} \in \mathcal{X}_2 \cap \mathcal{B}(x^*; \varepsilon_+^{\mu,\omega}(\delta))$, and $k \mapsto v(x_k)$ decreases for all k s.t. $x_k \notin \mathcal{X}_2 \cap \mathcal{B}(x^*; \varepsilon_-^{\mu,\omega}(\delta))$, yielding the required convergence. □

Remark 8. Observe that in the absence of an additive uncertainty, i.e. $\delta = 0$, (15) is automatically satisfied, while (16) yield $\varepsilon_-^{\mu,\omega}(0) = 0$ and $\varepsilon_+^{\mu,\omega}(0) = \frac{2\,(1-\mu)}{\omega}$. In that case, Newton's method (2), (12) converges to $\{x^*\}$, as per Theorem 2 and Remark 5. Alternatively, where the Hessian is known exactly, i.e. $\widetilde{H}(\cdot) = H(\cdot)$, $\mu \doteq 0$ may be selected in (14) and Theorem 7, yielding practical stability similarly to Theorem 4 with $\delta^H \doteq 0$, see Remark 5.

3. ROBUST STABILITY OF NEWTON'S METHOD

3.1 Input-to-state stability

It is convenient to again consider Newton's method in the presence of an additive disturbance, i.e. as per (2), (12), with $\widetilde{H} \equiv H$. In particular, consider

$$g(x, u) \doteq x - H(x)^{-1}\,\nabla f(x) + u, \tag{18}$$

for all $x, u \in \mathbb{R}^n$. For convenience, define $\mathscr{U}_\infty \doteq \ell_\infty^n$.

Input-to-state stability (ISS) is a robust stability property for nonlinear dynamical systems, first introduced by Sontag (1989). ISS provides a trajectory bound that scales with the norm of the initial state and input sequence. Specifically, a system (2) is input-to-state stable (ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$\|x(k, \xi, u)\| \leq \beta(\|\xi\|\,, k) + \gamma(\|u\|_\infty)$$

for all $\xi \in \mathbb{R}^n$, $u \in \mathscr{U}^\delta$, $\delta \in \mathbb{R}^2_{\geq 0}$, $k \in \mathbb{Z}_{\geq 0}$. ISS is compatible with Lyapunov theory, insofar as a Lyapunov characterization exists. In particular, system (2) is ISS if and only if there exists a continuous function $V : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$, called an ISS Lyapunov function, and α_1, α_2, $\alpha_3 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{19}$$

$$V(g(x, u)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|u\|), \tag{20}$$

for all $x \in \mathbb{R}^n$, $u \in \mathcal{U}$. This Lyapunov characterization can be used to demonstrate ISS of Newton's method (2), (18).

Theorem 9. Suppose there exist $\psi, \varphi \in \mathcal{K}_\infty$, $\kappa \in (0, 1)$ s.t.

- ⋆ $\psi - id \in \mathcal{K}_\infty$;
- ⋆ $\varphi(\lambda\,\|x\|) \leq \lambda\,\varphi(\|x\|)\ \forall\, \lambda \in (0, 1),\ x \in \mathbb{R}^n$;
- ⋆ $\psi(\|g(x, 0)\|) \leq \kappa\,\|x\|\ \forall\, x \in \mathbb{R}^n$. (21)

Then, $x \mapsto V(x) \doteq \varphi(\|x\|)$ is an ISS-Lyapunov function for Newton's method (2), (18), and hence it is ISS.

Proof. Fix $\psi, \varphi \in \mathcal{K}_\infty$, $\kappa \in (0, 1)$, as per the theorem statement, and fix any $x_0 \in \mathbb{R}^n$, $u \in \mathscr{U}_\infty$. Define $V(x) \doteq \varphi(\|x\|)$ for all $x \in \mathbb{R}^n$, and note that (19) holds with $\alpha_1 = \varphi = \alpha_2$. Define $\sigma \doteq \varphi \circ \psi \circ (\psi - id)^{-1} \in \mathcal{K}_\infty$. With $x_k \doteq x(k, x_0, u)$, observe by (18) that

$$\begin{aligned} &V(x_{k+1}) - V(x_k) = \varphi(\|g(x_k, 0) + u_k\|) - \varphi(\|x_k\|) \\ &\leq \varphi \circ \psi(\|g(x_k, 0)\|) + \varphi \circ \psi \circ (\psi - id)^{-1}(\|u_k\|) - \varphi(\|x_k\|) \\ &\leq \varphi(\kappa\,\|x_k\|) + \sigma(\|u_k\|) - \varphi(\|x_k\|) \\ &\leq -(1 - \kappa)\,\|x_k\| + \sigma(\|u_k\|), \end{aligned}$$

using the generalized triangle inequality for comparison functions, see (Kellett, 2014, Lemma 10, p. 347). Hence, (20) holds with $\alpha_3(s) \doteq (1 - \kappa)\,s$, $s \in \mathbb{R}_{\geq 0}$, as required. □

Remark 10. In the one dimensional case, i.e. $n \doteq 1$, consider polynomials of the form $f(x) \doteq \sum_{i=0}^N a_i\,x^{2\,i}$, $x \in \mathbb{R}$, given $a_i > 0$, $N \in \mathbb{Z}_{\geq 0}$ fixed. Straightforward calculations yield $\nabla f(x) = \left(2\,a_1 + \sum_{i=2}^N 2\,i\,a_i\,x^{2i-2} \right) x$,

$H(x) = 2\,a_1 + \sum_{i=2}^N 2\,i\,(2\,i-1)\,a_i\,x^{2i-2}$, so that $g(x,0) = \mu(x)\,x$, in which $\mu(x) \doteq \frac{\sum_{i=2}^N 2\,i\,(2\,i-2)\,a_i\,x^{2i-2}}{2\,a_1+\sum_{i=2}^N 2\,i\,(2\,i-1)\,a_i\,x^{2i-2}}$. Note further that there exists a $\kappa_0 \in (0,1)$ such that $|\mu(x)| \leq \kappa_0 < 1$ for all $x \in \mathbb{R}$, i.e. $\|g(x,0)\| \leq \kappa_0\,\|x\|$ for all $x \in \mathbb{R}$. Moreover, with $\psi(s) \doteq (1 + \frac{1-\kappa_0}{2\,\kappa_0})\,s$, $s \in \mathbb{R}_{\geq 0}$, note that $\psi - id \in \mathcal{K}_\infty$, and $\psi(\|g(x,0)\|) \leq \kappa\,\|x\|$ for all $x \in \mathbb{R}$, with $\kappa \doteq \kappa_0\,(1 + \frac{1-\kappa_0}{2\,\kappa_0}) = \frac{1+\kappa_0}{2} \in (0,1)$. That is, the first and third conditions in (21) hold.

Example 11. The polynomial on $\mathbb{R}$ defined by

$$f(x) \doteq x^4 + x^2 + 3$$

for all $x \in \mathbb{R}$ falls within the class defined by Remark 10. Select $\varphi, \psi \in \mathcal{K}_\infty$ with $\varphi(s) \doteq s^2$, $\psi(s) \doteq (\frac{11}{10})\,s$, $s \in \mathbb{R}_{\geq 0}$. By inspection, the first and second conditions in (21) hold. Meanwhile, as per Remark 10,

$$\psi\left(\|x - H^{-1}(x)\nabla f(x)\|\right) = \tfrac{11}{10}\,(1 - \tfrac{2x^2+1}{6x^2+1})\|x\| < \tfrac{11}{15}\,\|x\|,$$

for all $x \in \mathbb{R}$, so that the third condition in (21) holds. Hence, Theorem 9 implies that Newton's method (2), (18) is ISS.

Remark 12. There exist polynomials outside the class considered in Remark 10 for which Newton's method of (2), (18) remains ISS. For example, $f(x) \doteq x^4 + 4\,x^3 + 9\,x^2 + 1$, $x \in \mathbb{R}$, contains a cubic term that renders it inconsistent with Remark 10. However, $x \mapsto V(x) \doteq \|x\|^2$ is an ISS Lyapunov function for (2), (18).

3.2 Integral input-to-state stability

The type of robust stability implied by ISS may be too strong in specific applications of Newton's method, for example, where the disturbances involved are summable rather than uniformly bounded. Consequently, it is reasonable to consider weaker robust stability properties that may encapsulate the behaviour of Newton's method in the presence of such disturbances, where ISS does not hold. One such weaker robust stability property is integral ISS (iISS). A system (2) is iISS if there exist α, $\sigma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that

$$\alpha(\|x_k\|) \leq \beta(\|x_0\|, k) + \sum_{j=0}^{k-1} \sigma(\|u_j\|)$$

for all $x_0 \in \mathbb{R}^n$, $u \in \mathscr{U}^\delta \doteq \{u : \mathbb{Z}_{\geq 0} \to \mathbb{R} \mid \sum_{j=0}^{k-1} \sigma(\|u_j\|) \leq \delta\ \forall\ k \in \mathbb{Z}_{\geq 0}\}$, $\delta \in \mathbb{R}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 0}$.

As was the case with ISS, iISS is compatible with Lyapunov theory, i.e. a Lyapunov characterization exists. In particular, (2) is iISS if and only if there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, called an iISS Lyapunov function, and $\alpha_1, \alpha_2, \sigma \in \mathcal{K}_\infty$ and $\rho \in \mathcal{P}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{22}$$

$$V(g(x,u)) - V(x) \leq -\rho(\|x\|) + \sigma(\|u\|), \tag{23}$$

for all $x \in \mathbb{R}^n$, $u \in \mathcal{U}$. This Lyapunov characterization can be used to demonstrated iISS of Newton's method (2), (18). It is emphasized that iISS is a strictly weaker property than ISS, i.e. every system that is ISS is iISS, but not vice-versa. In particular, Newton's method may be iISS but not ISS.

Theorem 13. Suppose there exists $\rho \in \mathcal{P}$ such that

$$\|x\| - \|g(x,0)\| \geq \rho(\|x\|) \tag{24}$$

for all $x \in \mathbb{R}^n$. Then, Newton's method (2), (18) is iISS.

Proof. Let $\rho \in \mathcal{P}$ be as per the theorem statement, and define $V(x) \doteq \|x\| + \arctan(\|x\|)$ for all $x \in \mathbb{R}^n$. Define $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ by $\alpha_1(s) \doteq s$, $\alpha_2(s) \doteq s + \arctan(s)$, $s \in \mathbb{R}_{\geq 0}$, so that (22) holds. Fix any $x \in \mathbb{R}^n$, $u \in \mathbb{R}^n$. By definition of V, (18), and (24), note further that

$$\begin{aligned} &V(g(x,u)) - V(x) \\ &= \|g(x,u)\| - \|x\| + \arctan(\|g(x,u)\|) - \arctan(\|x\|) \\ &\leq \|g(x,0)\| - \|x\| + \|u\| \\ &\qquad + \arctan(\|g(x,0)\| + \|u\|) - \arctan(\|x\|) \\ &\leq -\rho(\|x\|) + \|u\| + \Delta(\|x\|, \|u\|), \end{aligned} \tag{25}$$

in which $\Delta(s,t) \doteq \arctan(s + t - \rho(s)) - \arctan(s)$, $s,t \in \mathbb{R}_{\geq 0}$. There are two cases involved in over-bounding $\Delta(s,t)$, namely $t - \rho(s) \geq 0$ and $t - \rho(s) < 0$. In the first case, Taylor's theorem implies that

$$t - \rho(s) \geq 0 \quad \Longrightarrow \quad \Delta(s,t) \leq t - \rho(s),$$

which implies via (25) that for $\|u\| - \rho(\|x\|) \geq 0$,

$$\begin{aligned} &V(g(x,u)) - V(x) \leq -\rho(\|x\|) + \|u\| + \|u\| - \rho(\|x\|) \\ &\quad = -2\,\rho(\|x\|) + 2\,\|u\| \leq -\rho(\|x\|) + 2\,\|u\|. \end{aligned} \tag{26}$$

Meanwhile, in the second case, $\arctan(s + t - \rho(s)) < \arctan(s)$, as $\arctan(\cdot)$ is strictly increasing, so that

$$t - \rho(s) < 0 \quad \Longrightarrow \quad \Delta(s,t) < 0,$$

which implies via (25) that for $\|u\| - \rho(\|x\|) < 0$,

$$V(g(x,u)) - V(x) < -\rho(\|x\|) + \|u\| \leq -\rho(\|x\|) + 2\,\|u\|.$$

which is as per (26). Hence, with $\sigma \in \mathcal{K}_\infty$ defined by $\sigma(s) \doteq 2\,s$, $s \in \mathbb{R}_{\geq 0}$, combining (25), (26) yields (23). That is, V satisfies (22), (23), yielding the claim. □

Example 14. Consider a function $f : \mathbb{R} \to \mathbb{R}$ implicitly defined via $\nabla f(x) = \exp\left(-\frac{\sqrt{x^2+1}+1}{2x^2}\right)\frac{x^{3/2}}{\sqrt{\sqrt{x^2+1}+1}}$, $x \in \mathbb{R}$, and $f(0) = 0$. The Hessian may be computed as $H(x) = \exp\left(-\frac{\sqrt{x^2+1}+1)}{2x^2}\right)\frac{\sqrt{\sqrt{x^2+1}+1}}{x^{3/2}}\sqrt{x^2+1}$, $x \in \mathbb{R}$. Hence,

$$\begin{aligned} &\|x\| - \|g(x,0)\| = \|x\| - \|x - H^{-1}(x)\nabla f(x)\| \\ &= \|x\| - \|x - x + \tfrac{x}{\sqrt{x^2+1}}\| \geq \tfrac{\|x\|}{x^2+1}\left(1 - \tfrac{1}{\sqrt{x^2+1}}\right). \end{aligned}$$

That is, (24) in Theorem 13 holds, with $\rho \in \mathcal{P}$ given by $\rho(s) \doteq \frac{s}{s^2+1}\left(1 - \frac{1}{\sqrt{s^2+1}}\right)$, for all $s \in \mathbb{R}_{\geq 0}$. Consequently, Newton's method (2), (18) is iISS by Theorem 13. Moreover, as $\rho \in \mathcal{P} \setminus \mathcal{K}$, note that it is not ISS.

Remark 15. Observe in Theorem 13 that if (24) holds with $\rho(s) \doteq \frac{1}{2}\,s$, $s \in \mathbb{R}_{\geq 0}$, i.e.

$$\|x - H(x)^{-1}\,\nabla f(x)\| \leq \tfrac{1}{2}\,\|x\|, \tag{27}$$

for all $x \in \mathbb{R}^n$, then $V(x) \doteq \log(\|x\| + 1)$, $x \in \mathbb{R}^n$, is an iISS-Lyapunov function. Since $\rho \in \mathcal{K}_\infty$, it follows immediately that Newton's method (2), (18) is also ISS.

Example 16. Consider $f : \mathbb{R}^3 \mapsto \mathbb{R}$ defined by

$$f(x) \doteq \tfrac{3}{5}\,x_1^{\frac{5}{3}} + \tfrac{1}{2}\,x_2^2 + \tfrac{4}{7}\,x_3^{\frac{7}{4}},$$

for all $x \in \mathbb{R}^3$. The gradient and Hessian are given by

$$\nabla f(x) = \begin{pmatrix} x_1^{\frac{2}{3}} \\ x_2 \\ x_3^{\frac{3}{4}} \end{pmatrix}, \quad H(x) = \begin{pmatrix} \frac{2}{3}\,x_1^{-\frac{1}{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4}\,x_3^{-\frac{1}{4}} \end{pmatrix}.$$

Consequently,

$$\left\|x - H^{-1}(x)\nabla f(x)\right\| = \left\|\left(-\tfrac{1}{2}\,x_1\ \ 0\ \ -\tfrac{1}{3}\,x_3\right)'\right\| \leq \tfrac{1}{2}\,\|x\|,$$

so that (27) holds. Hence, following Remark 15, Newton's method (2), (18) is ISS.

3.3 Incremental input-to-state-stability

Incremental input-to-state stability (δISS) describes a robust closeness of solutions property for dynamical systems that is consistent with ISS. It provides a means to assess sensitivity to initial conditions and controls. (It may be noted that a corresponding definition for incremental iISS is also available, but that property is not considered here.)

A system (2) is incrementally input-to-state stable (δISS) if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ such that any pair of trajectories of (2) satisfy $\|x(k, x_0, u) - x(k, y_0, v)\| \leq \beta(\|x_0 - y_0\|, k) + \gamma(\|u - v\|_\infty)$ for all $x_0, y_0 \in \mathbb{R}^n$, any $u, v \in \mathscr{U}^\delta$, $\delta \in \mathbb{R}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 0}$. Sufficient conditions for δISS for Newton's method (2), (7) follow using trajectory based arguments similar to those used for practical stability of (2), (7). To this end, given $L_1, L_2 \in \mathbb{R}_{>0}$ fixed, define

$$\mathcal{X}_3 \doteq \mathcal{B}(x^*; \delta_{\mathcal{X}}^L), \tag{28}$$

$$\delta_{\mathcal{X}}^L \doteq \sup \left\{ \hat{\delta} > 0 \;\middle|\; \begin{array}{c} \|H(x) - H(y)\| \leq L_1 \|x - y\| \\ \|\nabla f(x)\| \leq L_2 \\ \forall\, x, y \in \mathcal{B}(x^*; \hat{\delta}) \end{array} \right\}.$$

Theorem 17. Given $c \in (0, 1)$, $L_1, L_2 \in \mathbb{R}_{\geq 0}$ as per (4), (28), let $\delta \doteq (\delta^H, \delta^f) \in \mathbb{R}^2_{\geq 0}$ satisfy

$$\delta^H + \delta^f \leq \left(\tfrac{1 - K(\delta)}{K(\delta)}\right) \delta_{\mathcal{X}}^L, \quad K(\delta) < 1,$$

$$K(\delta) \doteq \max(c^2 L_1 L_2 + \max(c^2 L_1, c)\,(\delta^H + \delta^f), \; L_2 + \delta^H,\, c + \delta^f).$$

Then, Newton's method (2), (7) satisfies the δISS property for all $x_0, y_0 \in \mathcal{X}_3$, $u, v \in \mathscr{U}^\delta$.

Proof. Fix $c \in (0, 1)$, $L_1, L_2 \in \mathbb{R}_{\geq 0}$ as per (4), (28), let $\delta \doteq (\delta^H, \delta^f) \in \mathbb{R}^2_{\geq 0}$ as per the theorem statement.

Observe that $g : \mathbb{R}^n \times \mathcal{U}^\delta \to \mathbb{R}^n$ of (7) is Fréchet differentiable by the chain rule, using the norms $\|(h, v)\|_\sharp \doteq \|h\| + \|v\|_{\mathcal{U}}$ and $\|v\|_{\mathcal{U}} \doteq \|v^H\| + \|v^f\|$ for all $h \in \mathbb{R}^n$, $v \doteq (v^H, v^f) \in \mathcal{U}^\delta$, and

$$\begin{aligned} Dg(x, u)\,(h, v) &= D_x g(x, u)\, h + D_u g(x, u)\, v, \\ D_x g(x, u)\, h &= h - (D_x H(x)^{-1}\, h)\,(\nabla f(x) + u^f) \\ &\quad - (H(x)^{-1} + u^H) D_x \nabla f(x)\, h \\ &= -(D_x H(x)^{-1}\, h)\,(\nabla f(x) + u^f) - u^H\, H(x)\, h \\ &= -H(x)^{-1}(D_x H(x)\, h) H(x)^{-1}(\nabla f(x) + u^f) - u^H H(x) h, \\ D_u g(x, u)\, v &= -v^H\, \nabla f(x) - H(x)^{-1}\, v^f \; - 2 v^H v^f, \end{aligned}$$

for all $h \in \mathbb{R}^n$, $v \doteq (v^H, v^f) \in \mathcal{U}^\delta$. By (4), (28),

$$\begin{aligned} \|D_x g(x, u)\, h\| &\leq c^2 L_1 \|h\|\,(L_2 + \delta^f) + \delta^H c \|h\|, \\ &= (c^2 L_1 L_2 + \max(c^2 L_1, c)\,(\delta^H + \delta^f))\,\|h\| \\ \|D_u g(x, u)\, v\| &\leq \|v^H\|\, L_2 + c\,\|v^f\| \\ &\leq \max(L_2 + \delta^H, c + \delta^f)\,\|v\|_{\mathcal{U}}, \end{aligned}$$

so that $\|Dg(x, u)(h, v)\| = \|D_x g(x, u)\, h + D_u g(x, u)\, v\| \leq \|D_x g(x, u)\, h\| + \|D_u g(x, u)\, v\| \leq K\,\|(h, v)\|_\sharp$, where $K \doteq K(\delta) \doteq \max(c^2 L_1 L_2 + \max(c^2 L_1, c)\,(\delta^H + \delta^f),\, L_2 + \delta^H,\, c + \delta^f)$ as per the theorem statement. Hence, by the mean value theorem,

$$\begin{aligned} &\|g(y, v) - g(x, u)\| \leq \|(y - x, v - u)\|_\sharp \times \\ &\quad \int_0^1 \|Dg((x, u) + t(y - x, v - u))\|_{\mathscr{L}(\mathbb{R}^n \times \mathcal{U}^\delta; \mathbb{R}^n)}\, dt \\ &\leq K\,(\|y - x\| + \|v - u\|_{\mathcal{U}}). \end{aligned} \tag{29}$$

Fix arbitrary $x_0, y_0 \in \mathcal{X}_3$, $u, v \in \mathscr{U}^\delta$, and $x_k \doteq x(k, x_0, u)$, $y_k \doteq x(k, y_0, v)$, suppose that $x_k, y_k \in \mathcal{X}_3$, and set $\varepsilon_k \doteq \|x_k - y_k\|$, $x_{k+1} \doteq g(x_k, u_k)$, $y_{k+1} \doteq g(y_k, v_k)$. By (29),

$$\varepsilon_{k+1} \leq K\,(\varepsilon_k + \|v_k - u_k\|_{\mathcal{U}}) \leq K\,\varepsilon_k + K\,\|v - u\|_\infty.$$

Note in particular that for $u = 0$, $y_0 = x^*$, $y_k = x^*$, and

$$\begin{aligned} \|x_{k+1} - x^*\| &\leq K(\|x_k - x^*\|) + K\,(\delta^H + \delta^f) \\ &\leq K\,\delta_{\mathcal{X}}^L + K\,(\delta^H + \delta^f) \leq \delta_{\mathcal{X}}^L \end{aligned}$$

and so $x_{k+1} \in \mathcal{X}_3$. Similarly $y_{k+1} \in \mathcal{X}_3$. Hence,

$$\|x_k - y_k\| \leq K^k\,\|x_0 - y_0\| + \tfrac{1}{1-K}\,\|v - u\|_\infty$$

for all $k \in \mathbb{Z}_{\geq 0}$, which is the required δISS property. □

4. CONCLUSION

Sufficient conditions for practical stability, input-to-state-stability (ISS), iISS and incremental ISS were derived, and illustrated via simple examples.

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