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On sensor selection for differential algebraic systems observability

Sette Diop *

* *Laboratoire des Signaux & Systèmes, CNRS—CentraleSupélec—Université Paris-Sud, Université Paris-Saclay, Plateau de Moulon 3 rue Joliot Curie 91192 Gif sur Yvette cedex France, (e-mail: diop@l2s.centralesupelec.fr).*

Abstract: We address the problem of selecting sensors, that is, output equations, in order to endow dynamic equations of a system with some properties. Among all such desirable properties is the basic one of observability and/or identifiability. Once such a problem is solved one may ask how to choose sensors in order to improve estimation algorithms in terms of reliability, robustness, or simply, low complexity. First, what is the minimal number of sensors that make the dynamics observable? Second, when the sensors are bound to measure state components, what is their minimum number? Third, how may the observability margin be improved by selecting the sensors? In this communication we provide an overview of these questions in the differential algebraic approach of observation problems.

Keywords: Sensor selection, observability theory, observation problems, differential algebraic systems theory.

1. INTRODUCTION

Given state dynamics

$$\dot{x} = f(x, u) \tag{1}$$

with input u and state x there may be opportunity in practice for the designer to choose sensors, that is output equation, in order to endow the system with desirable properties in terms of observation problems. Selecting an output equation amounts to finding function

$$y = h(x, u) \tag{2}$$

such that the state becomes observable with respect to (u, y). This is part of various problems that are addressed in the litterature. In the parameter estimation context for linear systems, see for instance pioneering works Mehra (1976); Friedland (1977).

Notions of differential algebraic geometry used in this introduction are briefly recalled in a later section.

Let y be an arbitrary output of the system. By definition y is *componentwise algebraic* over $\mathbf{k}\langle u, x\rangle$. An output makes the dynamics observable if

$$\mathrm{d}^{\circ}_{\mathbf{k}\langle u,y\rangle}\mathbf{k}\langle u, x, y\rangle = 0\,,$$

that is, if each component of x is algebraic over $\mathbf{k}\langle u, y\rangle$.

Clearly, $y = x$ is an output which makes the system observable.

Let n denote the number of components of x, and

$$\mathbf{I} = \left\{p \in \mathbb{N} : \exists y_1, \cdots, y_p \in \mathbf{k}\left(u, x\right),\ \mathrm{d}^{\circ}_{\mathbf{k}\langle u,y\rangle}\mathbf{k}\langle u, x, y\rangle = 0\right\}.$$

The set $\mathbf{I}$ is the one of integers p such that there exists an output y with p components which makes the system observable.

It is a nonempty (since $n \in \mathbf{I}$) subset of $\mathbb{N}$. Therefore, $\mathbf{I}$ contains a smallest element which is precisely the *minimal number of sensors* which make the system observable.

To the author's opinion characterization of sensors minimal number is the first question which needs to be answered in sensor selection problem. And it seems to be an open problem. A very partial answer is given in a later section.

The next question is, when the output y is chosen as a *subset of the components of* x, instead of a vector rational function of u and x, what are those subsets of components of x which make the system observable.

In the differential algebraic approach of observation problems we may consider generalized implicit state dynamics

$$P(\dot{x}, x, \theta, \underline{u}) = 0 \tag{3}$$

where $\underline{u}$ designates a finite collection of derivatives of the input u, and P is a vector function with components P_i (non differential) polynomials in x, $\dot{x}$, θ and $\underline{u}$, and take output equation as polynomial equations

$$Q(y, x, \theta, \underline{u}) = 0 \tag{4}$$

and ask for an additional question: assume that we are given the freedom to select sensors among state components, how to choose them such that the system is identifiable, that is, each component of θ is algebraic over $\mathbf{k}\langle u, y\rangle$?

The differential algebraic approach of observation problems that is used here dates back to late eighties and early nineties with works of Pommaret (1986); Fliess (1987); Glad and Ljung (1990); Diop and Fliess (1991b,a). See Diop (2002) for a survey.

The main point of this approach, as first clarified in Diop and Fliess (1991b), is that a quantity, say z, of a system is observable with respect to some other one, say w (which is supposed to be available in some time interval), if each component of z is a solution of a (non differential) algebraic equation with coefficients eventually depending on w and finitely many of w's time derivatives.

The theory applies to models of systems in terms of *differential algebraic equations* only but which may be *implicit* in the variables to be observed.

In this approach the identifiability of constant parameters is simply viewed as the observability of these parameters in the system equations supplemented by differential equations expressing the fact that the time derivatives of the parameters are zero.

The remaining of this communication is organized as follows. Main lines of the differential algebraic approach of observability are recalled in the the next section. Then the single sensor selection theorem is recalled in section 3. Next we overview the sensor minimal number problem in section 4.

2. ON THE DIFFERENTIAL ALGEBRAIC APPROACH OF OBSERVATION PROBLEMS

A thorough introduction to the differential algebraic approach is available in Diop (2002). For the sake of completeness the following definition is recalled from there.

By a (differential algebraic) system with coefficients in $\mathbf{k}$ it is meant here a quasi-affine variety, $\mathcal{X}$ defined over $\mathbf{k}$. In other words, a system is defined by a set of differential equations together with a differential inequation.

A system is often specified along with its input, $u = (u_1, u_2, \ldots, u_m)$, output $y = (y_1, y_2, \ldots, y_p)$, and latent variable $z = (z_1, z_2, \ldots, z_\nu)$. Because the same differential equations usually define distinct systems when the input, output, and latent variable are specified in different ways.

But in this observability study, the crucial distinction between the system variables will be between the supposedly measured or known variables or quantities, and the those whose estimation, or observation is of interest. The variables of the system will, therefore be denoted by $\tau = (\tau_1, \tau_2, \ldots, \tau_\mu)$.

The differential Zariski closure of $\mathcal{X}$ is denoted by $\overline{\mathcal{X}}$, and the differential $\mathbf{k}$-algebra associated to the latter differential affine variety is denoted by $\mathbf{k}\left\{\overline{\mathcal{X}}\right\} = \mathbf{k}\left\{\tau\right\}$ and with coefficients in the ordinary differential field, $\mathbf{k}$, with characteristic zero. $\mathcal{X}$ is defined by giving differential quasi-affine variety in $\mathcal{U}^m \times \mathcal{U}^\nu \times \mathcal{U}^p$ where $\mathcal{U}$ stands for a differential universal field extension of $\mathbf{k}$.

With variables u, z, y, the differential $\mathbf{k}$-algebra associated to the latter differential affine variety is denoted by $\mathbf{k}\left\{\overline{\mathcal{X}}\right\} = \mathbf{k}\left\{u, z, y\right\}$ as explained below.

Let $\mu \in \mathbb{N}$, $\mu \geq 1$, and $\mathcal{U}\left\{T_1, T_2, \ldots, T_\mu\right\}$ be the differential polynomial $\mathcal{U}$-algebra in the differential indeterminates $T_1, T_2, \ldots, T_\mu$. If Σ is a subset of $\mathcal{U}\left\{T_1, T_2, \ldots, T_\mu\right\}$, $\mathbf{V}(\Sigma)$ denotes the subset of $\mathcal{U}^\mu$ consisting of the *zeros* of Σ in $\mathcal{U}^\mu$, i.e., the elements $(t_1, t_2, \ldots, t_\mu) \in \mathcal{U}^\mu$, such that $P(t_1, t_2, \ldots, t_\mu) = 0 \quad (P \in \Sigma)$. Conversely, if $\mathcal{X}$ is a subset of $\mathcal{U}^\mu$ then $\mathbf{I}(\mathcal{X})$ denotes the *defining differential ideal of* $\mathcal{X}$, i.e., the perfect differential ideal of $\mathcal{U}\left\{T_1, T_2, \ldots, T_\mu\right\}$ consisting of the differential polynomials P such that

$$P(t_1, t_2, \ldots, t_\mu) = 0 \quad ((t_1, t_2, \ldots, t_\mu) \in \mathcal{X}).$$

If a differential algebraic set $\mathcal{X}$ of $\mathcal{U}^\mu$ is defined over $\mathbf{k}$ then $\mathbf{I}(\mathcal{X})_{/\mathbf{k}}$ denotes the defining differential ideal of $\mathcal{X}$ over $\mathbf{k}$. For a differential algebraic set $\mathcal{X}$ which is defined over $\mathbf{k}$ $\mathcal{U}\left\{\mathcal{X}\right\}$ ($\mathbf{k}\left\{\mathcal{X}\right\}$, respectively) denotes the differential coordinate ring of $\mathcal{X}$ (the differential coordinate ring of $\mathcal{X}$ over $\mathbf{k}$, respectively), and $\mathcal{U}\langle\mathcal{X}\rangle$ ($\mathbf{k}\langle\mathcal{X}\rangle$, respectively) denotes the complete differential ring of quotients of $\mathcal{U}\left\{\mathcal{X}\right\}$ (the complete differential ring of quotients of $\mathbf{k}\left\{\mathcal{X}\right\}$ over $\mathbf{k}$, respectively) and call it the differential ring of differential rational functions on $\mathcal{X}$ (the differential ring of differential rational functions on $\mathcal{X}$ over $\mathbf{k}$, respectively). Recall that the differential algebra $\mathcal{U}\langle\mathcal{X}\rangle$ ($\mathbf{k}\langle\mathcal{X}\rangle$, respectively) is a field if, and only if, $\mathcal{X}$ is irreducible (irreducible over $\mathbf{k}$, respectively). An irreducible system $\mathcal{X}$ over a differential field $\mathbf{k}$ is equivalently defined by one of the three following data sets:

(i) *A positive integer* μ *and a set of differential polynomials,* $P_1, P_2, \ldots,$ *in* $\mathbf{k}\{T_1, \ldots, T_\mu\}$ *such that the perfect differential ideal generated by the* P_i*'s is prime.* The P_i's are usually said to be the *equations* of the system $\mathcal{X}$, but any (finite) set of generators of the perfect differential ideal, $\mathbf{I}(\mathcal{X})_{/\mathbf{k}} = \left\{P_1, P_2, \cdots\right\}_{/\mathbf{k}}$, of $\mathbf{k}\left\{T_1, T_2, \ldots, T_\mu\right\}$ which is generated by the P_i's play the same role. The *variables* of $\mathcal{X}$ are then the residues $\tau_1, \tau_2, \ldots, \tau_\mu$ of $T_1, T_2, \ldots, T_\mu$ $(\mathrm{mod}\ \mathbf{I}(\mathcal{X})_{/\mathbf{k}})$, respectively.

(ii) *A differential universal field* $\mathcal{U}$ *which is universal over* $\mathbf{k}$*, a positive integer* μ*, and an irreducible differential* $\mathbf{k}$*-algebraic set,* $\mathcal{X}$*, of* $\mathcal{U}^\mu$*.* The *equations* of the system $\mathcal{X}$ are then any set of generators of the defining differential ideal of the differential $\mathbf{k}$-algebraic set $\mathcal{X}$ over $\mathbf{k}$. The *variables* of $\mathcal{X}$ are the generic point, $\tau_1, \tau_2, \ldots, \tau_\mu$, consisting of the differential coordinate functions on the differential $\mathbf{k}$-algebraic set $\mathcal{X}$, but any other generic point, $\tau_1', \tau_2', \ldots, \tau_\mu'$, of the differential $\mathbf{k}$-algebraic set play the role of a new set of variables of the system $\mathcal{X}$, the corresponding equations being different from those which are relative to $\tau_1, \tau_2, \ldots, \tau_\mu$.

(iii) *A differential field* $\mathbf{k}$*-extension,* $\mathbf{k}\langle\mathcal{X}\rangle$*, differentially of finite type.* Given $\mathbf{k}\langle\mathcal{X}\rangle = \mathbf{k}\langle\tau_1, \tau_2, \ldots, \tau_\mu\rangle$, τ_1, $\tau_2, \ldots,$ τ_μ are the *variables* of $\mathcal{X}$. The *equations* of the system $\mathcal{X}$ are then any set of generators of the defining differential ideal of $\mathbf{k}\langle\tau_1, \tau_2, \ldots, \tau_\mu\rangle$, that is, the kernel of the differential morphism of differential $\mathbf{k}$-algebras of $\mathbf{k}\left\{T_1, T_2, \ldots, T_\mu\right\} \to \mathbf{k}\left\{\tau_1, \tau_2, \ldots, \tau_\mu\right\}$ which sends T_i into τ_i.

Variables of a system should be distinguished from values of these variables (called the *trajectories*) which verify the equations of the system. The above definitions of systems are incomplete in that the variables of a system are often partitioned into *external variables* (which consists of the input u (which itself consists of the control and/or the disturbance), and the output y), and the latent variables (that are denoted by z and which consists of the remainder of the variables of the system when the external variables are specified). The external variables are often attached to the system and cannot be arbitrarily changed without altering the definition of the system. *In particular, the input variable should be a differential transcendence basis of* $\mathbf{k}\langle\mathcal{X}\rangle$ *over* $\mathbf{k}$, *or at least, as containing such a differential transcendence basis.*

In observation problems, the system variable is partitioned into the *data*, or *observations*, $w = w_1, \ldots, w_\mu$, the variable being observed (or estimated) $z = z_1, \ldots, z_n$ and the remaining variables, ζ. In the classical observation problem, the data consist exclusively of (u, y), the control u and the measurements y. When the variable ζ is present, the projection $\mathcal{X}_{w,z}$ of $\mathcal{X}$ *along the variable* ζ is considered. It is the set of elements $(\overline{w}, \overline{z}) \in \overline{\mathbf{k}}^\mu \times \overline{\mathbf{k}}^n$ such that there is at least $\overline{\zeta}$ such that $(\overline{w}, \overline{z}, \overline{\zeta}) \in \mathcal{X}$.

In terms of equations, previously defined systems are those described by

$$\begin{cases} P_i(w, z, \zeta) = 0\,, \quad i = 1, 2, \ldots, \\ Q(w, z, \zeta) \neq 0\,, \end{cases} \tag{5}$$

where the P_i's and Q are finitely many polynomials in w, z, ζ and their derivatives.

For a system $\mathcal{X}$ the variable z is said to be *(algebraically) observable with respect to* w if the projection map $\pi : \mathcal{X}_{w,z} \to \mathcal{X}_w$ (sending every trajectory $(\overline{w}, \overline{z})$ of $\mathcal{X}_{w,z}$ onto the corresponding observation $\overline{w}$) is *generically finite*.

If z is observable with respect to w then the degree of π is called the *observability degree* of z with respect to w, and is denoted by $\mathrm{d}_w^\circ z$.

The variable z is said to be *rationally* observable with respect to w if it is observable with respect to w with observability degree one.

State systems of the form (1) are said to be *observable* if x is observable with respect to (u, y).

It was first proved in Diop and Fliess (1991b) (see Diop (2002) for more details) that the previous definition has a differential algebraic translation, namely: z is observable with respect to w iff z is algebraic over $\mathbf{k}\langle w\rangle$, that is, for each component, z_i of z there is a polynomial equation

$$H_i\,(z_i, w, \dot{w}, \ldots) = 0 \tag{6}$$

in z_i, and finitely many time derivatives of the data w, with coefficients in $\mathbf{k}$.

A quite general rank condition which applies to implicit differential algebaric systems has been obtained in Diop and Fliess (1991b). When specialized to rational dynamics (1) this rank condition is similar to (but is not formally the same as) the rank condition found in Hermann and Krener (1977).

The reader is referred to Diop (2002) for details on differential algebraic geometry terms or notations used here without explanations.

3. THE SINGLE SENSOR SELECTION THEOREM

When the output equation (2) is *scalar* and *linear* in x as follows

$$y = \alpha_1\, x_1 + \alpha_2\, x_2 + \cdots + \alpha_n\, x_n \tag{7}$$

then this rank condition reduces to the rank of the Wronskian matrix of the coefficients α. The existence of a scalar output (7) which makes the state of (1) observable then results from a classical theorem on the linear dependence of vectors of functions over constants.

In addition, the single sensor selection problem thus obtained implies that if the rational function f is with coefficients in a differential field of *constants* (say, $\mathbf{k} = \mathbb{R}$) then any set of functions of the time, $\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)$, which is linearly independent over $\mathbf{k}$, will make the state observable.

Theorem 1. Let the state dynamics of a rational state system $\mathcal{X}$ be given by

$$\dot{x} = f(x, u) \tag{8}$$

with a vector rational function f of the input u and the state x, and with coefficients in a differential field $\mathbf{k}$ of constants. Let m and n be the respective numbers of components of u and x. Let $\mathbf{K}$ be a differential extension field of $\mathbf{k}$ containing nonconstants. There always is a scalar output

$$y = \sum_{i=1}^{n} \alpha_i\, x_i \tag{9}$$

with α_1, $\alpha_2, \ldots, \alpha_n$ in $\mathbf{K}$, which makes x observable with respect to (u, y). Moreover, for y as in output 9 to make $\mathcal{X}$ observable it suffices that the associated α's be linearly independent over the subfield of constants of $\mathbf{K}$.

Remark 2. Note that the previous theorem is also an answer to the sensor selection identifiability problem mentionned in the introduction.

4. ON THE SENSOR MINIMAL NUMBER PROBLEM

Let us consider the sensor selection for the identifiability of θ in system 10.

$$\begin{cases} P_i(z, \theta, u) = 0\,, \quad i = 1, 2, \ldots, \\ Q(z, \theta, u) \neq 0\,, \end{cases} \tag{10}$$

In the previous definition of the set $\mathbf{I}$ the output was supposed to be in $\mathbf{k}\,(u, z)$, this may be relaxed by allowing components of y to be *algebraic* over $\mathbf{k}\,(u, z)$ instead.

$$\mathbf{I} = \Big\{p \in \mathbb{N} : \exists y_1, y_2, \cdots, y_p \text{ algebraic over } \mathbf{k}\,(u, z)\,, \\ \mathrm{d}^\circ_{\mathbf{k}\langle u,y\rangle}\mathbf{k}\langle u, \theta, y\rangle = 0\Big\}\,.$$

If the output y is chosen as a subset of the components of z, instead then a complete but *combinatorial* answer to the question is as follows.

Since the set of subsets of components of z is finite, they may be examined one to see if their measure is sufficient for θ to be observable with respect to (u, y). For each subset, $y = (z_{i_1} = z_1, \ldots, z_{i_p} = y_p)$, the test merely consists of computing the characteristic set of

$$\begin{cases} P_i(z, \theta, u) & (i \text{ finitely many}), \\ y_j = z_{i_j} & (1 \le j \le p) \end{cases}$$

with respect to a ranking such that any derivative of u and y is lower than any component of θ, whose derivatives, in turn, are lower than any state component which is not in y.

There are $2^n - 1$ observability tests thus to be done when the number of components of z is n!

The question thus reduces to *how to avoid most of those characteristic set computations, if possible?*

The answer to the question is positive if, and only if, $\theta_1, \theta_2, \ldots, \theta_q$ are algebraic over $\mathbf{k}\langle u, z\rangle$.

Let a ranking of $\mathbf{k}\{U, Z, \Theta\}$ be fixed such that every derivative of U is lower than Z; every derivative of Z is lower than Θ. Such a ranking is denoted as following.

$$\{u\}, \{z\}, \{\theta\}. \tag{11}$$

Let $\mathcal{A}$ be a characteristic set of $\mathcal{Z}$ with respect to ranking 11.

Lemma 3. $\mathcal{A}$ consists of two groups of differential polynomials:

$$\begin{cases} A_z(U, Z), \\ A_\theta(U, Z, \Theta). \end{cases}$$

$A_z(U, Z)$ represents the elements of $\mathcal{A}$ whose leaders are derivatives of components of Z. There are as many elements in this group as there are components of z. (Each component of Z has a derivative, necessarily, proper which is the leader of an element in $A_z(U, Z)$.) $A_\theta(U, Z, \Theta)$ consists of elements of $\mathcal{A}$ whose leaders are components of Θ. Each component of the parameter is the leader of a differential polynomial in $A_\theta(U, Z, \Theta)$.

From this lemma, it comes

Proposition 4. If p' designates the number of components of Z which are present in one of the elements of $A_\theta(U, Z, \Theta)$ (regardless the order of derivation) then

$$p \le p';$$

and it is sufficient to measure these p' components of z to guarantee the observability of θ with respect to (u, y).

To the question

$$p' \le p?$$

we make the following remarks. For the following system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 + x_2 \\ \dot{x}_3 = x_2 + x_3 \end{cases} \tag{12}$$

the minimal number of sensors is 1. The minimal polynomials of the latent variable are as follows

$$\ddot{x}_1 - \dot{x}_1 - x_1 = 0 \tag{13}$$

$$\ddot{x}_2 - \dot{x}_2 - x_2 = 0 \tag{14}$$

$$\dddot{x}_3 - 2\ddot{x}_3 + x_3 = 0 \tag{15}$$

In constrast, the minimal number of sensors for the system

$$\begin{cases} \dot{x}_1 = a\, x_1 \\ \dot{x}_2 = b\, x_2 \\ \dot{x}_3 = c\, x_3 \end{cases} \tag{16}$$

is 3. The minimal polynomials of the latent variable are as follows

$$\dot{x}_1 - a\, x_1 = 0\,,$$

$$\dot{x}_2 - b\, x_2 = 0\,,$$

$$\dot{x}_3 - c\, x_3 = 0\,.$$

But

$$y = \alpha\, x_1 + \beta\, x_2 + \delta\, x_3$$

makes the system observable as long as

$$\alpha\, \beta\, \delta (a - b)(b - c)(c - a) \neq 0\,.$$

It is not true that if the minimal number of sensors of a given dynamics is known to be 1 then the sensor may be placed at one of the state components. A counter-example may be exhibited from the previous remark.

5. CONCLUSION

Differential algebraic approach of observation problems seems to be well suited for the study of the sensor selection problem for the observability of systems though most questions are still unanswered. The relation to observability margin will be covered in the final version of this communication.

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