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Systemic risk and network intervention

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Abstract: We consider a novel adversarial shock/protection problem for a class of network equilibria models emerging from a variety of different fields as continuous network games, production networks, opinion dynamic models. The problem is casted into a min-max problem and analytically solved for two particular cases of aggregate performances: the mean square of the equilibrium or of its arithmetic mean. The main result is on the shape of the solutions, typically exhibiting a waterfilling type structure with the optimal protection concentrated in a proper subset of the nodes, depending significantly on the aggregate performance considered. The relation of the optimal protection with the Bonacich centrality is also considered.

Keywords: Network interventions, Systemic risk, Protections on network, Min-max problem, Bonacich centrality

1. INTRODUCTION

A key issue on network models is that of understanding how perturbations taking place at some nodes of a system propagate and possibly get amplified through the network interaction (systemic risk). Natural problems in this context are the classification of the network topologies on the basis of their systemic risk as well the individuation of the most fragile nodes in the network and the best policies to protect them from external shocks, potentially disruptive for the global system.

In this paper, we discuss such questions for a popular networked linear system that has recently appeared in several socio-economic models to represent Nash equilibria of a game or equilibrium configurations of a network opinion dynamics.

We consider a set of of n agents $\mathcal{V} = \{1, ..., n\}$ interacting through a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The strength of the interactions is determined by a matrix $A \in \mathbb{R}_+^{n\times n}$. We assume that $A_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$ and we interpret A_{ij} as a measure of how agent j influences agent i. We also equip every agent i with a scalar value $b_i \geq 0$ that is to be interpreted as a measure of resilience of the agent to the network interaction. Each agent i is stimulated by an input value $u_i \in \mathbb{R}$. The functionality of the overall system is described by a vector $x \in \mathbb{R}^n$ whose components describe the level of activity (state) of the single agents and satisfies the following balance equation

$$x_i = \sum_j A_{ij} x_j + b_i u_i, \quad i \in \mathcal{V} \tag{1}$$

In many applications, it is natural to assume that the weights are normalized so that $\sum_j A_{ij} + b_i = 1$ for every agent i. In this case, the relation (1) says that the state of each agent is a convex combination of its neighbors state and of its own input value and the weights are given by the rows of the matrix A and the resilience values b. Indicating with u the vector of the input values and with B the diagonal matrix having on the diagonal the elements b_i, we can rewrite (1) more compactly as

$$x = Ax + Bu \tag{2}$$

An instance of a concrete model leading to this relation is the popular Friedkin-Johnsen opinion dynamics model (Friedkin and Johnsen (1990))

$$x(k + 1) = Ax(k) + Bu \tag{3}$$

Here $x(k)$ represents the vector of the opinion hold by the various agents on a certain fact at time k. The opinion of each agent i evolves by taking a convex combination of the current neighbors opinion and its own input u_i. In this context, such input is referred to as anchor and typically represents the original belief of the agent. Notice that $B_{ii} = b_i$ indicates how much agent i is attached to its original belief and is thus resilient to the network interaction. Under the assumption that the spectral radius of A is strictly smaller than 1, the linear system (3) converges to the unique x satisfying relation (2).

Other examples where relation (2) naturally appears are in the description of Nash equilibria of continuous network games with linear best reply (Jackson and Zenou (2015),Bramoullé et al. (2014),Bramoullé and Kranton (2015)) and in the description of equilibria of production networks (Carvalho (2014),Acemoglu et al. (2010),Acemoglu et al. (2015)). In the economic interpretations, the components of the vector u can be interpreted as marginal benefits of the economic agents or productivity indices of firms.

In this paper, we consider the model described by (2) with the following standing assumptions:

- A is a sub-stochastic matrix (e.g. for all i, $\sum_j A_{ij} \leq 1$)

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- Indicating with $\rho(A)$ the spectral radius of A we must have $\rho(A) < 1$
- B is any diagonal matrix with non-negative elements

We don't assume, in general, the normalization relation $\sum_j A_{ij} + b_i = 1$ as we did in the previous example. As we will see later, in this way we will be able to cover other applicative contexts.

The spectral assumption allows to rewrite (2) as

$$x = (I - A)^{-1} Bu \tag{4}$$

A large attention has been recently devoted, particularly in the economic literature , to the effects that a perturbation in the vector u can have on the network equilibrium, in particular how shocks at the level of single agents can possibly be amplified by the network interaction, and propagate to the other agents. These studies have shown the role of the network topology (given by the pattern of non-zero elements of the matrix A) in determining the extent of these contagion phenomena. The performance typically considered in the literature is the mean squared error for x and for the aggregated function $n^{-1}\sum_i x_i$, the arithmetic mean of the equilibrium configuration.

In this paper, we take a further step in this direction and we analyze a more complex model, where shocks are complementary paired with protections, and we cast it into an adversarial min-max problem. Specifically, we assume that the vector u has the following structure:

$$u_i = \bar{u}_i + q_i^{-1}\eta_i$$

where $\bar{u}_i$ is a reference value, η_i is a random variable modeling the shock and $q_i \geq 1$ is the protection actuated on node i.

In this paper, we consider min-max optimization problems formulated in terms of the mean squared error of either x or the arithmetic mean $n^{-1}\sum_i x_i$ and we solve it under an (different) assumption on the correlations of the shocks. Solutions to these optimization problems in q will typically exhibit a 'waterfilling' structure with the optimal solution q concentrated on a limited number of nodes. The main message coming from our analysis is that the nodes on which protection has to be taken to minimize the effect of external shocks is not only function of the network topology but also depends on the correlation patterns of the shocks, as well on the type of functional we are considering.

We conclude presenting a brief outline of the rest of this paper. In Section 2 we formally introduce the min-max problems and in section 3 we review some motivating applications deriving from network games theory, economic network theory and opinion dynamics. In Section 4 we study the aforementioned problems in the particular case of independent shocks. In the last part we conclude with possible developments.

2. THE SHOCK-PROTECTION OPTIMIZATION PROBLEM

We consider the model described by (4) and we rewrite the assumption made on u in the following compact form:

$$u = \bar{u} + Q^{-1}\eta \tag{5}$$

where, precisely,

- $\bar{u}$ is a given reference vector;
- η is a random vector modeling the shock, having mean 0 and covariance matrix $\omega \in \Omega$;
- $Q = diag(q)$ where $q \in \mathcal{Q}$ is the *protection* vector.

We formulate the following min-max problems:

$$\min_{q\in\mathcal{Q}} \max_{\omega\in\Omega} \sum_i \mathrm{Var}[x_i] \tag{6}$$

$$\min_{q\in\mathcal{Q}} \max_{\omega\in\Omega} \mathrm{Var}[n^{-1}\mathbb{1}'x] \tag{7}$$

Ω and $\mathcal{Q}$ model the way the two adversarial can intervene in the system. For the sake of this paper, we have assumed that shocks are uncorrelated of total variance 1:

$$\Omega = \{\omega = diag(\sigma_1^2, ..., \sigma_n^2) \ \sum_i \sigma_i^2 = 1\} \tag{8}$$

The set $\mathcal{Q}$ is instead assumed to have the form

$$\mathcal{Q} = \{q \in \mathbb{R}_+^n \mid 1 \leq q_i \, \forall i, \ \|q\|_2 \leq C\} \tag{9}$$

The normalization of the power of the shocks to 1 is not restrictive as a different threshold can always be absorbed modifying the maximum protection norm C.

3. MOTIVATING EXAMPLES

We here present in detail some multi-agent models described by the equilibrium relation (4) starting with a coordination game strictly connected to the Friedkin-Johnsen model we previously revised.

3.1 Continuous coordination games and opinion dynamics models

We are given a set of players $\mathcal{V}$ whose utilities are given by

$$\mathcal{U}_i(x) = -\frac{1}{2}\left[\sum_j W_{ij}(x_i - x_j)^2 + \rho_i(x_i - u_i)^2\right]$$

where the coefficients W_{ij} and ρ_i are all non negative. This can be interpreted as a continuous variable coordination game with the presence of anchors u_i's.

Assume that $w_i = \sum_k W_{ik} > 0$ for every i and put

$$A_{ij} = \frac{W_{ij}}{w_i + \rho_i}, \ B_{ii} = \frac{\rho_i}{w_i + \rho_i}$$

If the set of nodes $\{i \in \mathcal{V} \,|\, u_i > 0\}$ is globally reachable, it can be shown that A has spectral radius less than 1 and that the game has just one Nash equilibrium given by formula (4) with A and B given above and u equal to the vector of anchors.

The Friedkin-Johnsen model (3) can be interpreted as a synchronous best response dynamics for this game.

3.2 Quadratic games

A variant of the games considered above are the quadratic games where utilities have the form

$$\mathcal{U}_i(x) = u_i x_i - \frac{1}{2}x_i^2 + \beta \sum_j W_{ij} x_i x_j \tag{10}$$

where the u_i's and β are positive constants and the elements W_{ij} are non negative. They are used to model socio-economic systems with complementarity effects (e.g. education, crime). The first two terms of (10) give the benefits and the costs to player i of providing the action level x_i. The last term instead reflects the vantage of cooperation of i with his friends (those j for which $W_{ij} > 0$). In the case when $\beta w_i < 1$ for all i (where, as above, $w_i = \sum_j W_{ij}$), we have that the unique Nash equilibrium is given by $x = (I - \beta W)^{-1}u$, a special case of (4).

3.3 Production networks

In the Cobb-Douglas model of an economy with firms producing distinct goods interconnected in the production process, the production outputs of the firms satisfy at equilibrium the following relation

$$x_i = \beta \sum P_{ij}x_j + (1-\beta)u_i \tag{11}$$

Here x_i has to be interpreted as the log of the output of firm i, P_{ij} as the share of the good produced by j in the production technology of firm i, u_i is the log productivity shock to firm i, and finally $\beta < 1$ is a constant indicating the level of interconnection in the economy. Relation (11) can be rewritten as $x = (I - \beta P)^{-1}(1-\beta)u$. In this model, a natural candidate for the economy's performance that is the real value added, can be expressed (in logarithm) simply as $y = n^{-1}\sum_i x_i$.

4. MAIN RESULTS

In this section, we give a complete solution to the optimization problems (6) and (7) under the standing assumption previously made of uncorrelated shocks, assuming that Ω and $\mathcal{Q}$ have the form (8) and (9), respectively.

In this case, the two cost functions can be rewritten as follows. We put $L = (I - A)^{-1}B$ and we compute

$$\begin{aligned}\sum_i \mathrm{Var}[x_i] &= \mathbb{E}[\eta' Q^{-1}L'LQ^{-1}\eta] \\ &= \mathrm{tr}(Q^{-1}\mathbb{E}[\eta\eta']Q^{-1}L'L) \\ &= \sum_i \left(\sigma_i \frac{\ell_i}{q_i}\right)^2\end{aligned} \tag{12}$$

where $\ell_i = \sqrt{(L'L)_{ii}}$ is the squared 2-norm of the i-th column of the matrix L.

Similarly, if we put $v = n^{-1}L'\mathbb{1}$, the variance of the arithmetic mean can be computed as

$$\mathrm{Var}[n^{-1}\mathbb{1}'x] = v'Q^{-1}\mathbb{E}[\eta\eta']Q^{-1}v = \sum_i \left(\sigma_i \frac{v_i}{q_i}\right)^2 \tag{13}$$

The two min-max problems can thus be framed under the general problem

$$\min_{\|q\|\le C}\ \max_{\|\sigma\|\le 1} \sum_i \left(\sigma_i \frac{y_i}{q_i}\right)^2, \tag{14}$$

where $y = \ell$ in the case of problem (6) and $y = v$ in the case of problem (7). We will refer to y_i as to the *centrality* of agent i and, without lack of generality, we assume that the elements of y are ordered in a decreasing order, i.e $y_1 \ge y_2 \ge ... \ge y_n$.

We now solve problem (14). First we introduce the function $f : (0, +\infty) \to \mathbb{R}$ given by

$$f(\lambda) = \sum_{i=1}^n \max\left\{1, \left(y_i/\sqrt{\lambda}\right)^2\right\}. \tag{15}$$

We notice that f is continuous, strictly decreasing in $(0, y_1^2]$, and

$$\lim_{\lambda\to 0+} f(\lambda) = +\infty, \quad f(y_1^2) = n.$$

This implies that for every $C \ge \sqrt{n}$, it is well defined $\lambda(C) := f^{-1}(C^2)$. We now define $k(C)$ as the maximum index such that $y_{k(C)} > \sqrt{\lambda(C)}$.

The following result holds true

Theorem 1. Let $C \ge \sqrt{n}$. It holds

$$\min_{\|q\|\le C}\ \max_{\|\sigma\|\le 1} \sum_i \left(\sigma_i \frac{y_i}{q_i}\right)^2 = \lambda(C) \tag{16}$$

and the optimum value for q is reached by

$$q_k = \begin{cases} y_k/\sqrt{\lambda(C)} & \text{if } k \le k(C) \\ 1 & \text{otherwise}\end{cases} \tag{17}$$

Proof.

Notice first that the internal maximization is solved by any vector σ concentrated on nodes i for which y_i/q_i is maximal. We can thus reformulate the problem as

$$\min_{\|q\|\le C}\ \max_i \left(y_i/q_i\right)^2. \tag{18}$$

and, introducing a new slack variable $\psi \in \mathbb{R}$, finally as

$$\begin{aligned}&\min_{\psi, \|q\|\le C}\ \psi \\ &\quad s.t.\ (y_i/q_i)^2 \le \psi,\ i = 1, ..., n.\end{aligned}$$

We introduce Lagrange multipliers $\alpha \in \mathbb{R}^n$, $\delta \in \mathbb{R}^n$, and γ and the Lagrangian function

$$\begin{aligned}L(q,\psi,\alpha,\delta,\gamma) = \psi + &\sum_{i=1}^n \alpha_i \left[(y_i/q_i)^2 - \psi\right] + \\ &\gamma\left(\sum_{i=1}^n q_i^2 - C^2\right) - \sum_{i=1}^n \delta_i (q_i - 1).\end{aligned}$$

Since the objective function and the constraints are convex and differentiable Karush-Khun-Tucker (KKT) conditions are necessary and sufficient for finding the optimum. Indicating with (q^*, ψ^*) and $(\alpha^*, \delta^*, \gamma^*)$ the optimum points for, respectively, the primal and the dual problem, KKT conditions are expressed as

$$q_i^* \ge 1, \quad \alpha_i^* \ge 0, \quad (y_i/q_i^*)^2 \le \psi^*, \quad i = 1, ..., n \tag{19}$$

$$\left((q_i^*)^2 - y_i\sqrt{\alpha_i^*/\gamma^*}\right)(q_i^* - 1) = 0, \qquad i = 1, ..., n \tag{20}$$

$$\alpha_i^*\left[(y_i/q_i^*)^2 - \psi^*\right] = 0, \qquad i = 1, ..., n \tag{21}$$

$$\|q^*\|_2 = C, \tag{22}$$

$$(q_i^*)^2 - y_i\sqrt{\alpha_i^*/\gamma^*} \ge 0, \qquad i = 1, ..., n \tag{23}$$

$$1 - \sum_{i=1}^n \alpha_i^* = 0, \qquad i = 1, ..., n. \tag{24}$$

Note that, if for some i, $\psi^* < y_i^2$, then, by (19), $q_i^* > 1$. From (20) we deduce that $\alpha_i^* \neq 0$ and (21) finally yields $q_i^* = y_i^2/\psi^*$. If instead $\psi^* \ge y_i^2$, then $q_i^* = 1$. In fact, if $q_i^* > 1$ then, from (21) it would follow that $\alpha_i^* = 0$, and, substituting in (20), we would get a contradiction.

Therefore, for every i, $q_i^* = \max\{1, y_i/\sqrt{\psi^*}\}$. Plugging these values in (21) we obtain $f(\psi^*) = C^2$ or, equivalently, $\psi^* = \lambda(C)$. This proves the result. □

We now comment on the result obtained.

Remark 2. The structure of the optimum we have found shows how the optimal protection is in general concentrated on a proper subset of nodes. In this respect, it is interesting to analyze various regimes depending on the chosen budget cost C.

(1) Notice that protection is active in just one node, namely $q_k = 1$ for all $k > 2$, if and only if $y_2 < \sqrt{\lambda(C)}$ or, equivalently,

$$C < \sqrt{f(y_2^2)} = \sqrt{n + y_1^2/y_2^2 - 1}$$

In this case, we get from (15) that the optimal value is given by

$$\lambda(C) = y_1^2/(C^2 - n + 1) \tag{25}$$

We will refer to this as to the *low budget regime*.

(2) Notice that protection is active on all nodes, namely $q_k > 1$ for all k, if and only if $y_n > \sqrt{\lambda(C)}$ or, equivalently,

$$C > \sqrt{f(y_n^2)} = \|y\|/y_n \tag{26}$$

We will refer to this as to the *high budget regime*. In this regime, we get from (15) that the optimal value si given by

$$\lambda(C) = \|y\|^2/C^2 \tag{27}$$

(3) It follows from the shape of the optimum (17) that when the protection is active $q_i > 1$, then the level of protection q_i is proportional to the centrality of the node y_i.

5. EXAMPLES

We now go back to our original problems and present a number of examples. Clearly, the optimal protection solutions to the two optimization problems (12) and (13) in the measure that the two centrality vectors, either ℓ for the minimization of the mean square error of x or v for the minimization of the mean square error of $n^{-1}\mathbb{1}'x$, may differ. In the following we denote by q_v^* and q_ℓ^* the optimal protection solutions related to the two problems.

For the sake of simplicity we consider a special case of our general network equilibrium problem, already met in the motivating examples:

$$x = (1-\beta)(I - \beta P)^{-1}u \tag{28}$$

where P is a stochastic matrix and $\beta \in (0, 1)$. It corresponds to the model for the production output of a networked economy reviewed in Subsection 3.3. It can also be seen as the Nash equilibrium of a coordination game in the special case when the strength of the anchors relatively to the network interactions strength is the same for all nodes.

In this case, particularly in the economic applications, the matrix $L = (1-\beta)(I-\beta P)^{-1}$ (that is also stochastic) takes the name of *Leontief* matrix. The vector $v = n^{-1}L'\mathbb{1}$ is a stochastic vector, known as the *Bonacich centrality* of the network described by P.

For this problem, we study the behavior of q_v^* and q_ℓ^* for three different networks: a star undirected graph, a directed graph with 11 nodes, and a real dataset of a production network with 417 node.

Example 1. Consider the undirected star graph S_{n+1} with $n + 1$ nodes. Denoted by W its adjacency matrix, we put $P_{ij} = b_i^{-1}W_{ij}$ where $b_i = \sum_j W_{ij}$ is the degree of node i. Notice that for symmetric reasons, all quantities (v_i, ℓ_i, $(q_v^*)_i$, $(q_\ell^*)_i$) will be the same for all leaves i in the star. We use subscripts 0 and ε to indicate any component relative, respectively, to the center of the star or to the leaves.

Values of v and ℓ are:

$$v_0 = \frac{1 + \beta(n-1)}{n(1+\beta)}, \quad v_\varepsilon = \frac{\beta + (n-1)}{n(n-1)(1+\beta)}$$

$$\ell_0 = \left(\frac{1 + \beta^2(n-1)}{(1+\beta)^2}\right)^{1/2}$$

$$\ell_\varepsilon = \left(\frac{\beta^2 + (n-1)\left[2(2\beta^2 - \beta^4) + n(1-\beta^2)^2 - 1\right]}{(n-1)^2(1+\beta)^2}\right)^{1/2}.$$

For such family of graphs, there will be just one threshold describing the behavior of each optimal solution as shown in (26): $\|v\|/v_\epsilon$ and $\|\ell\|/\ell_\epsilon$. When C is below this threshold, the corresponding q^* (either q_v^* or q_ℓ^*) is concentrated in just the center of the star, when is above, it is instead diffused also on the leaves. Computing such thresholds for large n, we obtain that

$$\|v\|/v_\epsilon \sim \beta n, \quad \|\ell\|/\ell_\epsilon \sim \gamma\sqrt{n}$$

where

$$\gamma = \frac{(1 + 2\beta^2 - 2\beta)^{1/2}}{1+\beta}$$

This significative difference between the two thresholds indicates that there are going to be three different regimes for the budget with a central large regime for which the optimal protection for the mean square error of x is diffused, while the one for the mean square error of $n^{-1}\mathbb{1}'x$ is instead concentrated on the center of the star. This is intuitive as in the aggregate performance $n^{-1}\mathbb{1}'x$, shocks at leaves level have a smaller impact because of the cancellation induced by averaging. In table 1 we report, in the large scale limit, the details of the protection vectors q_v^* and q_ℓ^* in the three regimes.

Table 1. Optimal protections q_v^* and q_ℓ^* for star graph when $n \to \infty$.

	$C \le \gamma\sqrt{n}$	$C \in \left(\gamma\sqrt{n}, \beta n\right)$	$C \ge \beta n$
$q_{v_0}^*$	$\sqrt{C^2-n}$	$\sqrt{C^2-n}$	C
$q_{v_\varepsilon}^*$	1	1	$C/(\beta n)$
$q_{\ell_0}^*$	$\sqrt{C^2-n}$	$C\sqrt{1-1/\gamma^2}$	$C\sqrt{1-1/\gamma^2}$
$q_{\ell_\varepsilon}^*$	1	$C/\left(\gamma\sqrt{n}\right)$	$C/\left(\gamma\sqrt{n}\right)$

Example 2. Consider the directed network of $n = 11$ agents in figure 1.

As in the previous example we take P as the normalized adjacency matrix $P_{ij} = b_i^{-1}W_{ij}$. In this example we have considered $\beta = 0.58$.

First three columns of table 2 represent, respectively, indexes of nodes, vector v, and vector ℓ for the network in the figure 1. Fourth and fifth columns represent, respectively, the values of q_v^* and q_ℓ^* for a fixed budget $C = 3.962$. These

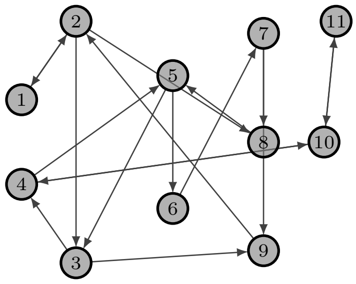

Fig. 1. Directed network of $n = 11$ nodes.

values highlight, also in this case an important difference between the two optimal solutions: notice that q_v^* is greater than 1 only for 6 agents while q_ℓ^* is greater than 1 for all the agents.

Table 2. Optimal protections q_v^* and q_ℓ^*.

i	v	ℓ	q_v^*	q_ℓ^*
1	0.0575	0.4730	1.0000	1.0294
2	0.1334	0.6965	1.6485	1.5159
3	0.0901	0.4941	1.1137	1.0754
4	0.0950	0.5428	1.1734	1.1813
5	0.1124	0.5577	1.3891	1.2138
6	0.0708	0.4632	1.0000	1.0081
7	0.0792	0.5104	1.0000	1.1109
8	0.0805	0.4775	1.0000	1.0393
9	0.1066	0.5521	1.3178	1.2016
10	0.1056	0.6749	1.3054	1.4689
11	0.0688	0.5425	1.0000	1.1807

Figure 2 shows entries of q_v^* (in blue) and q_ℓ^* (in red) as a function of C from $C = \sqrt{n}$ to $C = n$. There is a flat line until the high regime is not reached.

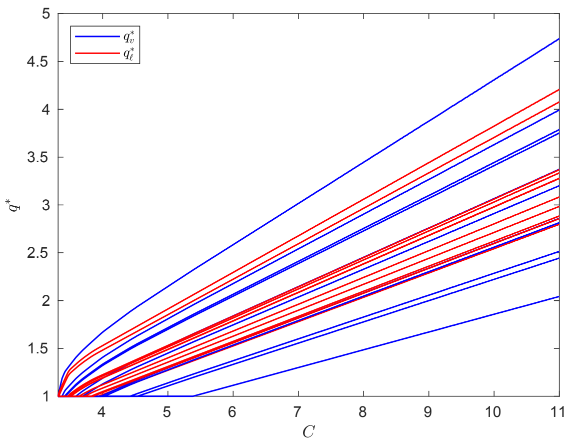

Fig. 2. Entries of optimal protections q_v^* and q_ℓ^* as a function of the budget C. Origin is located in $(\sqrt{n}, 1)$.

Example 3. In this example we consider a network derived from a concrete dataset describing production interconnections among US firms published in 2002 by the Bureau of Economic Analysis. This dataset has been used in (Acemoglu et al. (2012)) to which we refer for further description. The model consists of an input-output matrix W (Commodity-by-Commodity Direct Requirements Tables) that represents direct interactions between $n = 417$ commodities. The value W_{ij} gives the input share of commodity j used in the production of commodity i. P is the normalized version of it that is used in analyzing production networks with constant returns to scale technologies.

We have set $\beta = 0.58$ taking this value from the aforementioned article.

For this example we have plot in figures 3 and 5 the optimal value $\lambda(C)^*$ and we have compared it against the protection extended to all nodes in a way proportional to their centrality. We indicate such value $\lambda(C)^{\text{diff}}$. Figures 4 and 6 represent the ratio between the strategies.

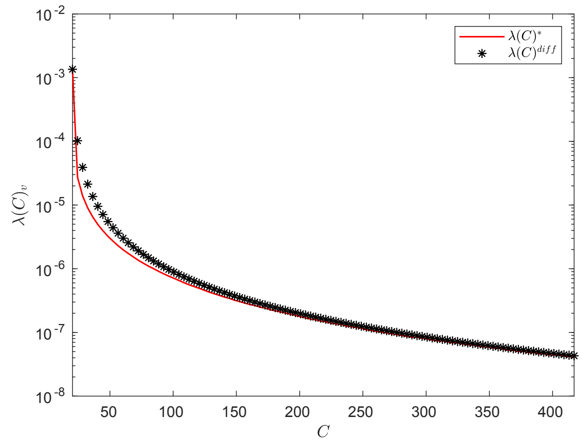

Fig. 3. Optimal variance of the arithmetic mean $\lambda(C)_v^*$ compared with $\lambda(C)_v^{\text{diff}}$.

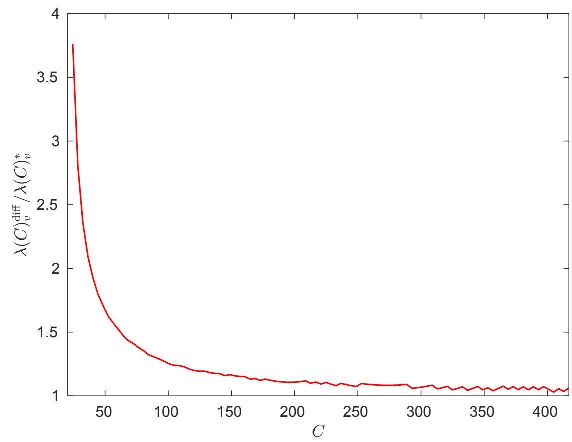

Fig. 4. Ratio between $\lambda(C)_v^*$ and $\lambda(C)_v^{\text{diff}}$ as function of C.

6. CONCLUSION AND FUTURE DIRECTIONS

We have presented a new class of problems where shocks are complementary paired with protections and casted into an adversarial min-max problem. We have explicitly solved the optimization problem, for the case of two possible aggregative performance functionals, and studied the form of the optimal protection vector that exhibits a waterfiling shape that crucially depends on the aggregate performance chosen.

Future research steps will be in the direction of considering correlated shocks and more realistic constraints in the protection vector (e.g. heterogeneity of nodes).

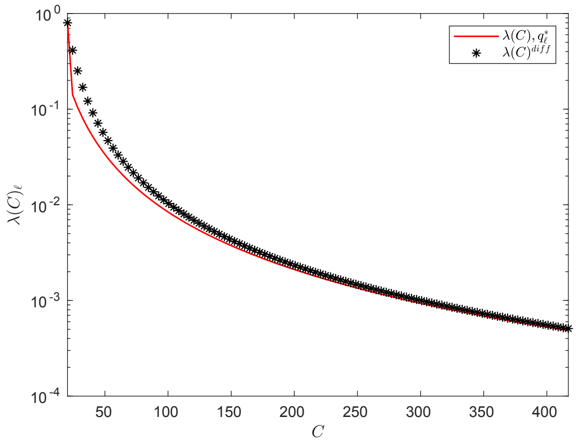

Fig. 5. Optimal variance of x $\lambda(C)_\ell^*$ compared with $\lambda(C)_\ell^{\text{diff}}$.

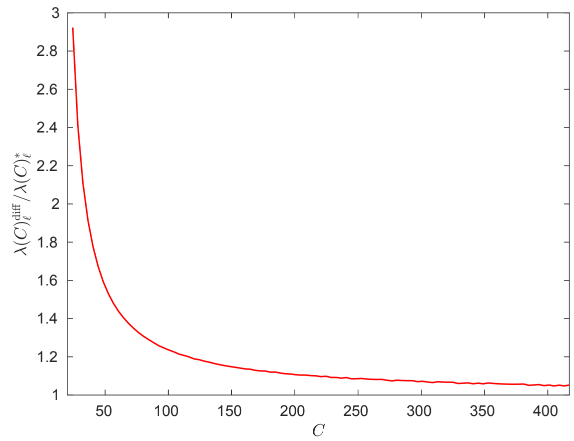

Fig. 6. Ratio between $\lambda(C)_\ell^*$ and $\lambda(C)_\ell^{\text{diff}}$ as function of C.

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