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Zonotopic set-membership estimation for Switched Systems based on W_i-Radius Minimization: Vehicle application

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Abstract: This work is devoted to the problem of guaranteed set-membership state estimation based on zonotopes for discrete-time switched systems with bounded uncertainties and unknown inputs. The additive uncertainties in this contribution are treated using the so called W_i-Radius. The size of the zonotope which contains the real system state is decreased at each sample time. The complete solution procedure is formulated as a LMI-constrained optimization problem. The proposed design is applied to the robust state estimation of vehicle lateral dynamics state. Simulation results based on real data demonstrate the performance of the proposed method.

Keywords: Interval observer, state estimation, vehicle lateral dynamics, Linear Matrix

1. INTRODUCTION

The set-membership techniques have been intensively studied during last decades and applied to state estimation of uncertain dynamical systems [Ifqir et al. 2018, Schweppe 1968, Le et al. 2012, Combastel 2015]. In this method, a set containing all admissible state trajectories that are consistent with the measured output and the norm-bounded uncertainty is provided. No other assumptions on the distribution of the uncertainties are needed. Different geometrical forms have been used to represent this set as ellipsoids [Schweppe 1968], zonotopes [Combastel 2015], interval boxes [Ifqir et al. 2018] and polytopes [Chisci et al. 1998]. In this paper, a geometrical representation based on zonotopes is used considering its good tradeoff between the estimation accuracy and computational complexity. A zonotope is a linear transformation of an unitary box. They have been used in Alamo et al. [2005] for guaranteed state estimation of nonlinear discrete-time systems and in Wang et al. [2016, 2017] for differential-algebraic-equation (DAE) and discrete-time descriptor systems. Unfortunately, and to the best of authors knowledge, such study has not been reported earlier in the literature for switched dynamical systems.

Motivated by the above observation, this note presents a new approach for guaranteed state estimation for the case of uncertain switched discrete-time systems subject to known and unknown inputs. A bounded description of the state disturbances and measurement noise is considered. The main result is an algorithm to compute a set that contains the unknown state trajectories which are consistent with the measured output and the given bounds of disturbances and noise. The consistency test is implemented by finding a parameterized intersection zonotope between the measurement state set and the predicted state set. An optimization methodology based on linear matrix inequalities (LMIs) is proposed to minimize the radius of the obtained zonotope. The effectiveness of the proposed methodology is illustrated through an application to vehicle sideslip angle estimation. The vehicle lateral dynamics model used in this study is based on the known bicycle model [Rajamani 2011] associated with a vision system measurement. Real data recorded on a vehicle test are used to validate the proposed algorithm. However, it should be noticed that the use of the zonotopic set-membership technique is a novelty in the context of vehicle dynamics estimation. The obtained results have been successfully demonstrated in this new area of application.

The remainder of the paper is organized as follows. First, some preliminary definitions and the problem formulation are introduced in Section 2. The zonotopic guaranteed set-membership approach based on the W_i-Radius minimization is presented in Section 3. Experimental results of applying the proposed set-membership approach to estimate the vehicle sideslip angle are shown in Section 4. Finally, conclusions are drawn in Section 5.

2. PRELIMINARIES AND PROBLEM FORMULATION

Before presenting the main result of this paper, some preliminary definitions and properties are briefly introduced.

2.1 Notations and basic definitions

Throughout the paper, the set of all natural and real numbers will be denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. $\mathbb{R}^n$ will denote the n-dimensional real vector space. A^T will refer to the transpose of a matrix A. $A > 0$ (resp. $A < 0$) denotes

a matrix with positive (resp. negative) components, $A \succ 0$ (resp. $A \prec 0$) denotes the positive (resp. negative) definite matrix. $\mathcal{I}_n$ stands for the identity matrix of dimension $n \times n$ and $\oplus$ to the Minkowski sum. The symbol $(*)$ will be used to denote the symmetric terms in symmetric block matrices.
Given a center vector $p \in \mathbb{R}^n$ and a matrix $H \in \mathbb{R}^{n\times m}$, the Minkowski sum of the segments defined by the columns of a matrix H is called *zonotope* of order m and will be defined by $\mathcal{X} = \langle p, H\rangle = p \oplus H\mathbf{B}^m = \{p + Hz : z \in \mathbf{B}^m\}$ where $\mathbf{B}^m$ stands for an unitary box composed by m unitary intervals.

Given a zonotope $\mathcal{X} = \langle p, H\rangle \subset \mathbb{R}^n$ and a weighting matrix $W_i \in \mathbb{R}^{n\times n}$, the W_i-Radius of $\mathcal{X}$ is defined by $l^W = \max_{z\in\mathcal{X}} \|z - p\|^2_{W_i} = \max_{b\in\mathbf{B}^r} \|Hb\|^2_{W_i}$.

Property 1. [Le et al. 2013] The Minkowski sum of two zonotopes $\mathcal{X}_1 = p_1 \oplus H_1\mathbf{B}^{m_1}$ and $\mathcal{X}_2 = p_2 \oplus H_2\mathbf{B}^{m_2}$ is also a zonotope defined by $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 = (p_1 + p_2) \oplus [H_1\ H_2]\mathbf{B}^{m_1+m_2}$.

The interval hull of a zonotope $\mathcal{X}$ (i.e. the smallest interval box that contains $\mathcal{X}$) will be denoted by $\mathcal{X}^\star$.

Property 2. [Combastel 2015] For a zonotope $\mathcal{X} \subset \mathbb{R}^n$ with $H \in \mathbb{R}^{n\times m}$, its interval hull, $\mathcal{X}^\star = [a,\ b]$, is obtained as follows:

$$\begin{aligned} a_i &= p_i - \sum_{j=1}^{m} |H_{i,j}|, \quad i = 1, \dots, n \\ b_i &= p_i + \sum_{j=1}^{m} |H_{i,j}|, \quad i = 1, \dots, n \end{aligned} \tag{1}$$

Property 3. [Combastel 2015] A reduction operator, denoted $\downarrow_{q,W}$, allows to reduce the number of generators of a zonotope $\mathcal{X}$ to a fixed number $q \geq n$, such that $\mathcal{X} = \langle p,\ H\rangle \subset \langle p, \downarrow_{q,W}(H)\rangle$. The procedure for implementing the operator $\downarrow_{q,W}$ is summarized as follows:

(1) Sort the column of segment matrix $H \in \mathbb{R}^{n\times m}$ in decreasing weighted vector norm $\|.\|_W$, $H = [h_1, \dots, h_j, \dots, h_m]$, $\|h_j\|^2_W \geq \|h_{j+1}\|^2_W$;
(2) Enclose the set $H_>$ generated by the $m-q+n$ smaller columns into a box (i.e., interval hull):
If $m \leq q$, **then** $\downarrow_{q,W}(H) = \downarrow_W(H)$,
Else $\downarrow_{q,W}(H) = [H_>,\ rs(H_<)] \in \mathbb{R}^{n\times q}$
$H_> = [h_1, \dots, h_{q-n}]$, $H_< = [h_{q-n+1}], \dots, h_m]$

where $rs(H_<)$ is a diagonal matrix with diagonal elements of $rs(H_<)_{i,i} = \sum_{j=1}^m |H_{<_{i,j}}|,\ \ i = 1, \dots, n$.

2.2 Problem statement

Consider the discrete-time switched model described by the following equations:

$$\begin{aligned} x_{k+1} &= A_{\sigma(k)}x_k + B_{\sigma(k)}u_k + E_{\sigma(k)}d_k + \omega_{\sigma(k)} \\ y_k &= C_{\sigma(k)}x_k + \upsilon_{\sigma(k)} \end{aligned} \tag{2}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, $d_k \in \mathbb{R}^{n_d}$, $y_k \in \mathbb{R}^{n_y}$ denote the state vector, known and unknown inputs and measurement output, respectively. $A_{\sigma(k)} \in \mathbb{R}^{n_x\times n_x}$, $B_{\sigma(k)} \in \mathbb{R}^{n_x\times n_u}$, $E_{\sigma(k)} \in \mathbb{R}^{n_x\times n_d}$ and $C_{\sigma(k)} \in \mathbb{R}^{n_y\times n_x}$ are state, input, unknown input and observation matrices. $\sigma(k)$ is a piecewise-constant function, called the *switching signal*, assumed to be known and takes its values in the finite set $\mathcal{I} = \{1, \dots, N\}$, where $N > 1$ is the number of subsystems. Before we proceed with the detailed set-estimator design, the following assumptions are considered since they are used throughout the paper.

Assumption 1. The initial state is assumed to be unknown but bounded by a zonotope $\mathcal{X}_0 = \langle p_0, H_0\rangle$, where $p_0 \in \mathbb{R}^{n_x}$ and $H_0 \in \mathbb{R}^{n_x\times n_x}$ are the center and segment matrix of this zonotope.

Assumption 2. The disturbance vector $\omega_{\sigma(k)} \in \mathbb{R}^{n_\omega}$ and measurement noise vector $\upsilon_{\sigma(k)} \in \mathbb{R}^{n_\upsilon}$ are assumed to be unknown but bounded by zonotopes $\mathcal{W}_{\sigma(k)} = \langle 0, D_{\sigma(k)}\rangle$ and $\mathcal{V}_{\sigma(k)} = \langle 0, F_{\sigma(k)}\rangle$, respectively.

Assumption 3. For the switched system (2), let us assume that the following rank condition is satisfied $\forall\sigma(k),\ \ k \in \mathbb{N}$:

$$rank(C_{\sigma(k)}E_{\sigma(k)}) = rank(E_{\sigma(k)}) = n_d \tag{3}$$

Thus, there exists a non-empty set of solutions of switched matrices $P_{\sigma(k)}$ and $M_{\sigma(k)}$ satisfying $\forall\sigma(k)$

$$P_{\sigma(k)} + M_{\sigma(k)}C_{\sigma(k)} = \mathcal{I}_{n_x}, \quad P_{\sigma(k)}E_{\sigma(k)} = 0 \tag{4}$$

Some preliminary definitions are introduced before stating the main results.

Definition 1. (*Uncertain state set*). Given the switched system (2) with $x_0 \in \mathcal{X}_0$, $\omega_{\sigma(k)} \in \mathcal{W}_{\sigma(k)}$, $\forall\sigma(k)$ and for all $k \in \mathbb{N}$, the uncertain state set $\overline{\mathcal{X}}_k$ is defined by

$$\begin{aligned} \overline{\mathcal{X}}_k = \Big\{ & x \in \mathbb{R}^{n_x} \Big| x \in A_{\sigma(k)}\overline{\mathcal{X}}_{k-1} \oplus B_{\sigma(k)}u_{k-1} \oplus \\ & D_{\sigma(k)}\mathcal{W}_{\sigma(k)} \oplus E_{\sigma(k)}d_{k-1} \Big\} \end{aligned} \tag{5}$$

Definition 2. (*Measurement state set*). Given the switched system (2), a measurement output vector y_k and $\upsilon_k \in \mathcal{V}_{\sigma(k)}$, $\forall\sigma(k)$ and for all $k \in \mathbb{N}$, the measurement state set $\mathcal{P}(k)$ is defined by

$$\mathcal{P}(k) = \left\{ x \in \mathbb{R}^{n_x} \middle| C_{\sigma(k)}x_k - y_k = F_{\sigma(k)}s_1, \forall s_1 \in \mathbf{B}^{n_\upsilon} \right\} \tag{6}$$

Definition 3. (*Exact uncertain set*). Given the switched system (2), a measurement output vector y_k, $\omega_{\sigma(k)} \in \mathcal{W}_{\sigma(k)}$, $\upsilon_k \in \mathcal{V}_{\sigma(k)}$, $\forall\sigma(k)$ and for all $k \in \mathbb{N}$, the exact uncertain state set $\mathcal{X}(k)$ is defined by

$$\mathcal{X}(k) = \overline{\mathcal{X}}_k \cap \mathcal{P}(k) \tag{7}$$

The goal is to approximate the exact uncertain set $\mathcal{X}(k)$ by an outer approximation of the intersection between the uncertain trajectory and the region of the state space that is consistent with the measured output y_k and the initial state set $\mathcal{X}_0$.
Let us assume that $x_k \in \mathcal{X}(k) \subseteq \hat{\mathcal{X}} = \langle \hat{p}, \hat{H}_k\rangle$ at time $k \in \mathbb{N}$ that also satisfies $x_0 \in \mathcal{X}_0$ at time $k = 0$. Suppose also that a measured output y_k is obtained at time instant k. Under these assumptions, an outer bound of the exact uncertain state set $\mathcal{X}(k)$ can be estimated using the following algorithm.
Step i *(Prediction step)*: Given the switched system (2), compute the zonotope $\overline{\mathcal{X}}_{k+1}$ that bounds the set of predicted states for the uncertain trajectory of the system;
Step ii *(Measurement step)*: Compute the measurement state set $\mathcal{P}_{k+1}$ by using the measurement vector y_{k+1};
Step iii *(Correction step)*: To find the state estimation set, compute an outer approximation $\hat{\mathcal{X}}_{k+1}$ of the intersection between $\overline{\mathcal{X}}_{k+1}$ and $\mathcal{P}_{k+1}$.

3. GUARANTEED STATE INTERSECTION

In what follows, a set-membership state estimation approach based on zonotopes for switched discrete-time system (2) is proposed. This approach is based on *parameterized intersection zonotope* for implementing the measurement consistency test.

The aim is to find a parameterized zonotope $\hat{\mathcal{X}}_{k+1}$ that contains the intersection of the two sets $\overline{\mathcal{X}}_{k+1}$ and $\mathcal{P}_{k+1}$ used in the previous algorithm. The zonotope $\hat{\mathcal{X}}_{k+1}$ is parameterized with respect to a switched correction matrix $\Lambda_{\sigma(k)} \in \mathbb{R}^{n_x \times n_y}$.

Theorem 1. Given the switched system (2), a measurement output vector y_{k+1}, $x_0 \in \mathcal{X}_0$, $\omega_{\sigma(k)} \in \mathcal{W}_{\sigma(k)}$, $v_k \in \mathcal{V}_{\sigma(k)}$, $\forall \sigma(k)$, $x_k \in \langle \hat{p}_k, \hat{H}_k \rangle \subseteq \langle \hat{p}_k, \overline{H}_k \rangle$ with $\overline{H}_k = \downarrow_{q,W} (\hat{H}_k)$, $P_{\sigma(k)} \in \mathbb{R}^{n_x \times n_x}$ and $M_{\sigma(k)} \in \mathbb{R}^{n_x \times n_y}$ satisfying (4). Then, for any switched correction matrix $\Lambda_{\sigma(k)} \in \mathbb{R}^{n_x \times n_y}$, $x_{k+1} \in \left\{ \overline{\mathcal{X}}_{k+1} \cap \mathcal{P}_{k+1} \right\} \subseteq \hat{\mathcal{X}}_{k+1} = \langle \hat{p}_{k+1}, \hat{H}_{k+1} \rangle$, where

$$\begin{aligned} \hat{p}_{k+1} &= T_{\sigma(k)} P_{\sigma(k)} A_{\sigma(k)} \hat{p}_k + T_{\sigma(k)} P_{\sigma(k)} B_{\sigma(k)} u_k \\ &+ (M_{\sigma(k)} + \Lambda_{\sigma(k)} - \Lambda_{\sigma(k)} C_{\sigma(k)} M_{\sigma(k)}) y_{k+1} \end{aligned} \tag{8a}$$

$$\hat{H}_{k+1} = \begin{bmatrix} T_{\sigma(k)} P_{\sigma(k)} A_{\sigma(k)} \overline{H}_k \\ T_{\sigma(k)} P_{\sigma(k)} D_{\sigma(k)} \\ T_{\sigma(k)} M_{\sigma(k)} F_{\sigma(k)} \\ \Lambda_{\sigma(k)} F_{\sigma(k)} \end{bmatrix}^T \tag{8b}$$

where $T_{\sigma(k)} = \mathcal{I}_{n_x} - \Lambda_{\sigma(k)} C_{\sigma(k)}$.

Proof. For any $x_{k+1} \in \left\{ \hat{\mathcal{X}}_{k+1} \cap \mathcal{P}_{k+1} \right\}$, we know $x_{k+1} \in \hat{\mathcal{X}}_{k+1}$ and $x_{k+1} \in \mathcal{P}_{k+1}$. Consider the switched system (2) with the inclusion $x_k \in \langle \hat{p}_k, \hat{H}_k \rangle \subseteq \langle \hat{p}_k, \overline{H}_k \rangle$ and $\omega_{\sigma(k)} \in \mathcal{W}_{\sigma(k)}$, there exists a vector $s_2 \in \mathbf{B}^{q+n_\omega}$ such that

$$x_{k+1} = A_{\sigma(k)} \hat{p}_k + B_{\sigma(k)} u_k + E_{\sigma(k)} d_k + \left[A_{\sigma(k)} \hat{H}_k,\ D_{\sigma(k)} \right] s_2 \tag{9}$$

Besides, from $x_{k+1} \in \mathcal{P}_{k+1}$, there exists a vector $s_1 \in \mathbf{B}^{n_v}$ such that

$$C_{\sigma(k)} x_{k+1} - y_{k+1} = F_{\sigma(k)} s_1 \tag{10}$$

Considering a pair of switched matrices $P_{\sigma(k)}$ and $M_{\sigma(k)}$ satisfying (4) $\forall \sigma(k)$, combining (2) and (10) leads to

$$\begin{aligned} &(P_{\sigma(k)} + M_{\sigma(k)} C_{\sigma(k)}) x_{k+1} = P_{\sigma(k)} A_{\sigma(k)} \hat{p}_k + \\ &P_{\sigma(k)} B_{\sigma(k)} u_k + P_{\sigma(k)} E_{\sigma(k)} d_k + M_{\sigma(k)} y_{k+1} + \\ &\left[P_{\sigma(k)} A_{\sigma(k)} \hat{H}_k \ \ P_{\sigma(k)} D_{\sigma(k)} \right] s_2 + M_{\sigma(k)} F_{\sigma(k)} s_1 \end{aligned} \tag{11}$$

Set $R_{\sigma(k)} = \left[P_{\sigma(k)} A_{\sigma(k)} \hat{H}_k \ \ P_{\sigma(k)} D_{\sigma(k)} \ \ M_{\sigma(k)} F_{\sigma(k)} \right]$ and $s = \left[s_1^T,\ s_2^T \right]^T$. If the matrix equations (4) hold, then (11) can be simplified as follows

$$\begin{aligned} x_{k+1} &= P_{\sigma(k)} A_{\sigma(k)} \hat{p}_k + P_{\sigma(k)} B_{\sigma(k)} u_k \\ &+ M_{\sigma(k)} y_{k+1} + R_{\sigma(k)} s \end{aligned} \tag{12}$$

Let $\Lambda_{\sigma(k)} \in \mathbb{R}^{n_x \times n_x}$, by adding and substituting a correction term $\Lambda_{\sigma(k)} C_{\sigma(k)} R_{\sigma(k)} \beta$ in (12), we obtain $\forall \sigma(k)$

$$\begin{aligned} x_{k+1} &= P_{\sigma(k)} A_{\sigma(k)} \hat{p}_k + P_{\sigma(k)} B_{\sigma(k)} u_k + M_{\sigma(k)} y_{k+1} \\ &+ \Lambda_{\sigma(k)} C_{\sigma(k)} R_{\sigma(k)} \beta + (\mathcal{I}_{n_x} - \Lambda_{\sigma(k)} C_{\sigma(k)}) R_{\sigma(k)} s \end{aligned} \tag{13}$$

By substituting x_{k+1} in (10) by (12), we have

$$\begin{aligned} &C_{\sigma(k)} R_{\sigma(k)} s = y_{k+1} - C_{\sigma(k)} M_{\sigma(k)} y_{k+1} - \\ &C_{\sigma(k)} P_{\sigma(k)} A_{\sigma(k)} \hat{p}_k - C_{\sigma(k)} P_{\sigma(k)} B_{\sigma(k)} u_k + F_{\sigma(k)} s_1 \end{aligned} \tag{14}$$

then, by replacing $C_{\sigma(k)} R_{\sigma(k)} \beta$ in (13), it follows that

$$\begin{aligned} &x_{k+1} = (\mathcal{I}_{n_x} - \Lambda_{\sigma(k)} C_{\sigma(k)}) P_{\sigma(k)} A_{\sigma(k)} \hat{p}_k + \\ &(\mathcal{I}_{n_x} - \Lambda_{\sigma(k)} C_{\sigma(k)}) P_{\sigma(k)} B_{\sigma(k)} u_k + \\ &(M_{\sigma(k)} + \Lambda_{\sigma(k)} - \Lambda_{\sigma(k)} C_{\sigma(k)} M_{\sigma(k)}) y_{k+1} + \\ &\left[(\mathcal{I}_{n_x} - \Lambda_{\sigma(k)} C_{\sigma(k)}) R_{\sigma(k)},\ \Lambda_{\sigma(k)} F_{\sigma(k)} \right] \begin{bmatrix} s \\ s_1 \end{bmatrix} \end{aligned} \tag{15}$$

Thus, by using the above definition of $R_{\sigma(k)}$ and by letting $T_{\sigma(k)} = \mathcal{I}_{n_x} - \Lambda_{\sigma(k)} C_{\sigma(k)}$, (8a) and (8b) are obtained and the proof is complete. ■

Remark 1. Recall that, the zonotope reduction operator $\downarrow_{q,W} (.)$ defined in Property 3 is used in order to reduce the computational complexity during the state propagation.

The size of the intersection zonotope $\hat{\mathcal{X}}_{k+1}$ can be measured by the W_i-radius as follows

$$\begin{aligned} l_{k+1}^W &= \max_{x_{k+1} \in \hat{\mathcal{X}}_{k+1}} \left\| x_{k+1} - \hat{p}_{k+1}(\Lambda_{\sigma(k)}) \right\|_{2, W_{\sigma(k)}}^2 \\ &= \max_{z \in \mathbf{B}^{(q+n_x+2n_y)}} \left\| \hat{H}_{k+1}(\Lambda_{\sigma(k)}) z \right\|_{2, W_{\sigma(k)}}^2 \\ &= \max_{z \in \mathbf{B}^{(q+n_x+2n_y)}} z^T \hat{H}_{k+1}^T(\Lambda_{\sigma(k)}) W_{\sigma(k)} \hat{H}_{k+1}(\Lambda_{\sigma(k)}) z \end{aligned} \tag{16}$$

where $W_{\sigma(k)} = W_i$, $\forall i \in \{1, \ldots, N\}$ is the weighting matrix of appropriate dimensions for the i-th subsystem. Note that the design of the correction matrix $\Lambda_{\sigma(k)}$ is required to minimize the effects of uncertainties and guarantee that the size of the intersection zonotope is not increasing. Then, if there exists a scalars $\alpha_{\sigma(k)}$ and $\gamma_{\sigma(k)}$ associated with each subsystem $\sigma(k) = i$ such that

$$l_{k+1}^W \leq (1 - \alpha_{\sigma(k)}) l_k^W + \gamma_{\sigma(k)} \epsilon_{\sigma(k)} \tag{17}$$

where $\epsilon_{\sigma(k)}$ is a positive switched constant that represents the max influence of disturbances and measurement noises as follows:

$$\epsilon_{\sigma(k)} = \max_{s_1 \in \mathbf{B}^{n_\omega}} \left\| D_{\sigma(k)} s_1 \right\|_2^2 + \max_{s_2 \in \mathbf{B}^{n_v}} \left\| F_{\sigma(k)} s_2 \right\|_2^2 \tag{18}$$

the size of $\hat{\mathcal{X}}_{k+1}$ is decreasing. If (17) holds, then for time instant $k \to \infty$, this expression is equivalent for all $i \in \{1, \ldots, N\}$ to

$$l_\infty^W = (1 - \alpha_i) l_\infty^W + \gamma_i \epsilon_i \tag{19}$$

it follows that

$$l_\infty^W = \frac{\gamma_i \epsilon_i}{\alpha_i},\ \forall i \in \{1, \ldots, N\} \tag{20}$$

Let us consider the i-th ellipsoid $\mathcal{E}_i \triangleq \left\{ x \mid x^T W_i x < \frac{\gamma_i \epsilon_i}{\alpha_i}, \forall i \in \{1, \ldots, N\} \right\}$ which can be normalized as follows: $\mathcal{E}_i \triangleq \left\{ x \mid x^T \frac{\alpha_i W_i}{\gamma_i \epsilon_i} x < 1, \forall i \in \{1, \ldots, N\} \right\}$. This ellipsoid is related to the W_i-radius of the guaranteed zonotopic state estimation at infinity. To minimize the W_i-radius (i.e l_∞^W), the smallest diameter of the ellipsoid is sought by solving an Eigenvalue Problem (EVP) [Boyd et al. 1994]. Therefore, there exists a positive switched scalar τ_i such that

$$\begin{aligned} &\tau_i > 0,\ \ \forall i \in \{1, \ldots, N\} \\ &\frac{\alpha_i W_i}{\gamma_i \epsilon_i} \succeq \tau_i \mathcal{I}_{n_x},\ \ \forall i \in \{1, \ldots, N\} \end{aligned} \tag{21}$$

Then, the design of the correction matrix Λ_i associated with each subsystem i can be done by solving a LMI (Linear Matrix Inequality) optimization problem with the following constraints:

Theorem 2. Given the intersection zonotope $\mathcal{X}_{k+1} = \langle \hat{p}_{k+1}, \hat{H}_{k+1} \rangle$ in (8), (17) holds if there exists a matrix $Y_i \in \mathbb{R}^{n_x \times n_y}$, a positive definite matrix $W_i \in \mathbb{R}^{n_x \times n_x}$, for given scalars $\alpha_i \in (0, 1)$, $\gamma >$ and $\epsilon_i > 0$ such that the following LMI problem holds

$$\begin{aligned} &\max_{\tau_i, W_i, Y_i} \tau_i \\ &\tau_i > 0,\ \ \forall i \in \{1, \ldots, N\} \end{aligned} \tag{22}$$

$$\frac{\alpha_i W_i}{\gamma_i \epsilon_i} \succeq \tau_i \mathcal{I}_{n_x}, \quad \forall i \in \{1, \dots, N\} \tag{23}$$

with $Y_i = W_i \Lambda_i$.

Proof. Let $\Delta_k^{l_W} = l_{k+1}^W - l_k^W$, then using (16) we have

$$\Delta_k^{l_W} = \max_{z \in \mathbf{B}^{(q+n_x+2n_y)}} \left\| \hat{H}_{k+1}(\Lambda_{\sigma(k)}) z \right\|_{2,W_{\sigma(k)}}^2 - \max_{\hat{z} \in \mathbf{B}^q} \left\| \overline{H}_k(\Lambda_{\sigma(k)}) \hat{z} \right\|_{2,W_{\sigma(k)}}^2 \tag{25}$$

Let $z = \begin{bmatrix} \overline{z}^T & s_1^T & s_2^T & s_3^T \end{bmatrix}^T \in \mathbf{B}^{q+n_x+2n_y}$ with $\overline{z} \in \mathbf{B}^q$, $s_1 \in \mathbf{B}^{n_x}$, $s_2 \in \mathbf{B}^{n_y}$ and $s_3 \in \mathbf{B}^{n_y}$. Since $\max_{\hat{z} \in \mathbf{B}^q} \|\overline{H}_k(\Lambda_{\sigma(k)})\hat{z}\|_{2,W_{\sigma(k)}}^2 \geq \|\overline{H}_k(\Lambda_{\sigma(k)})\overline{z}\|_{2,W_{\sigma(k)}}^2$, $\forall\, \overline{z} \in \mathbf{B}^q$, we obtain from (25)

$$\begin{aligned}
\Delta_k^{l_W} &= \max_{\overline{z} \in \mathbf{B}^q, s_1 \in \mathbf{B}^{n_x}, s_2 \in \mathbf{B}^{n_y}, s_3 \in \mathbf{B}^{n_y}} \left(\left\| \hat{H}_{k+1}(\Lambda_{\sigma(k)}) z \right\|_{2,W_{\sigma(k)}}^2 - \left\| \overline{H}_k(\Lambda_{\sigma(k)}) \overline{z} \right\|_{2,W_{\sigma(k)}}^2 \right) \\
&= \max_{\overline{z} \in \mathbf{B}^q, s_1 \in \mathbf{B}^{n_x}, s_2 \in \mathbf{B}^{n_y}, s_3 \in \mathbf{B}^{n_y}} \left(\begin{bmatrix} \overline{z} \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}^T \hat{H}_{k+1}^T(\Lambda_{\sigma(k)}) \times W_{\sigma(k)} \hat{H}_{k+1}(\Lambda_{\sigma(k)}) \begin{bmatrix} \overline{z} \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} - \overline{z}^T \overline{H}_k^T W_{\sigma(k)} \overline{H}_k \overline{z} \right)
\end{aligned} \tag{26}$$

By adding and subtracting the terms $\max_{\overline{z} \in \mathbf{B}^q} \alpha_{\sigma(k)} \left\| \overline{H}_k \overline{z} \right\|_{2,W_{\sigma(k)}}^2$ and $-\gamma_{\sigma(k)} \epsilon_{\sigma(k)}$ where $\epsilon_{\sigma(k)}$ is given by (18), (26) is rewritten as

$$\begin{aligned}
\Delta_k^{l_W} = &\max_{\overline{z} \in \mathbf{B}^q, s_1 \in \mathbf{B}^{n_x}, s_2 \in \mathbf{B}^{n_y}, s_3 \in \mathbf{B}^{n_y}} \left(\begin{bmatrix} \overline{z} \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}^T \hat{H}_{k+1}^T(\Lambda_{\sigma(k)}) \times W_{\sigma(k)} \hat{H}_{k+1}(\Lambda_{\sigma(k)}) \begin{bmatrix} \overline{z} \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} + \alpha_{\sigma(k)} \overline{z}^T \overline{H}_k^T W_{\sigma(k)} \overline{H}_k \overline{z} - \right. \\
&\left. \overline{z}^T \overline{H}_k^T W_{\sigma(k)} \overline{H}_k \overline{z} - \gamma_{\sigma(k)} s_1^T D_{\sigma(k)}^T D_{\sigma(k)} s_1 - \gamma_{\sigma(k)} s_2^T F_{\sigma(k)}^T F_{\sigma(k)} s_2 \right) - \max_{\overline{z} \in \mathbf{B}^q} \alpha_{\sigma(k)} \left\| \overline{H}_k \overline{z} \right\|_{2,W_{\sigma(k)}}^2 + \gamma_{\sigma(k)} \epsilon_{\sigma(k)}
\end{aligned} \tag{27}$$

Substituting (8b) into (27), we get

$$\begin{aligned}
&\max_{\overline{z} \in \mathbf{B}^q, s_1 \in \mathbf{B}^{n_x}, s_2 \in \mathbf{B}^{n_y}, s_3 \in \mathbf{B}^{n_y}} \left(\begin{bmatrix} \overline{H}_k \overline{z} \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}^T \left(\Theta_{\sigma(k)}^T W_{\sigma(k)} \Theta_{\sigma(k)} + \begin{bmatrix} (\alpha_{\sigma(k)} - 1) W_{\sigma(k)} & 0 & 0 & 0 \\ 0 & -\gamma_{\sigma(k)} D_{\sigma(k)}^T D_{\sigma(k)} & 0 & 0 \\ 0 & 0 & -\gamma_{\sigma(k)} F_{\sigma(k)}^T F_{\sigma(k)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \right. \\
&\left. \begin{bmatrix} \overline{H}_k \overline{z} \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} \right) - \max_{\overline{z} \in \mathbf{B}^q} \alpha_{\sigma(k)} \left\| \overline{H}_k \overline{z} \right\|_{2,W_{\sigma(k)}}^2 + \gamma_{\sigma(k)} \epsilon_{\sigma(k)}
\end{aligned} \tag{28}$$

where $\Theta_{\sigma(k)} = \begin{bmatrix} T_{\sigma(k)} P_{\sigma(k)} A_{\sigma(k)} \\ T_{\sigma(k)} P_{\sigma(k)} D_{\sigma(k)} \\ T_{\sigma(k)} M_{\sigma(k)} F_{\sigma(k)} \\ \Lambda_{\sigma(k)} F_{\sigma(k)} \end{bmatrix}$. If the following inequality holds $\forall i \in \mathcal{I}$

$$\begin{bmatrix} (\alpha_{\sigma(k)} - 1) W_{\sigma(k)} & 0 & 0 & 0 \\ 0 & -\gamma_{\sigma(k)} D_{\sigma(k)}^T D_{\sigma(k)} & 0 & 0 \\ 0 & 0 & -\gamma_{\sigma(k)} F_{\sigma(k)}^T F_{\sigma(k)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \Theta_{\sigma(k)}^T W_{\sigma(k)} \Theta_{\sigma(k)} + \prec 0 \tag{29}$$

which is equivalent by Schur complement to (24), then

$$\Delta_k^{l_W} \leq -\alpha_{\sigma(k)} l_k^W + \gamma_{\sigma(k)} \epsilon_{\sigma(k)} \tag{30}$$

from which we prove that satisfying (24) leads to satisfy the condition (17). Furthermore, the tight size of the intersection zonotope must be sought, hence the introduction of the conditions (22) and (23) which completes the proof of Theorem 2. ■

4. APPLICATION TO VEHICLE LATERAL DYNAMICS ESTIMATION

4.1 Vehicle model description

The model used in this study describes vehicle lateral dynamics in a cornering lane, which is obtained from the known bicycle model and the relative vehicle angular and lateral displacements in the road trajectory (Fig. 1) [Rajamani 2011].

The two degrees of freedom model describing the vehicle yaw and lateral motions can be represented by the following equations:

$$m v_x (\dot{\beta} + r) = F_{yf} + F_{yr}, \quad I_z \dot{r} = l_f F_{yf} - l_r F_{yr}. \tag{31}$$

where the description of vehicle dynamics parameters is given in Appendix A.

Using the so-called Pacejka magic formula [Pacejka and Bakker 1992], and under assumption of small sideslip angle variations, lateral forces F_{yf} and F_{yr} are considered to be linear and expressed as follows:

$$F_{yf} = c_f (\delta_f - \beta - \frac{lf}{v_x} r), \; F_{yr} = c_r (-\beta + \frac{lr}{v_x} r) \tag{32}$$

In the proposed model, it is assumed that the available measurements are yaw rate r, longitudinal velocity v_x and front steering angle δ_f. Gathering equations (31) and (32) and chosing β and r, as state variables, leads to the following state equations:

$$\begin{bmatrix} \dot{\beta} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} -\dfrac{c_f + c_r}{m v_x} & \dfrac{c_r l_r - c_f l_f}{m v_x^2} - 1 \\ \dfrac{c_r l_r - c_f l_f}{I_z} & -\dfrac{c_r l_r^2 + c_f l_f^2}{I_z v_x} \end{bmatrix} \begin{bmatrix} \beta \\ r \end{bmatrix} + \begin{bmatrix} \dfrac{c_f}{m v_x} \\ \dfrac{c_f l_f}{I z} \end{bmatrix} \delta_f \tag{33}$$

We consider also a vision system measuring the vehicle angular and lateral displacements at a look-ahead distance l_s. The equations describing the evolution of the measurements obtained using a vision camera are given by

$$\dot{\psi}_L = r - v_x \rho, \;\; \dot{y}_L = v_y + v_x \psi_L + l_s (r - v_x \rho) \tag{34}$$

The system (34) can be rewritten in the following state representation form:

$$\begin{bmatrix} \dot{\psi}_L \\ \dot{y}_L \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & v_x \end{bmatrix} \begin{bmatrix} \psi_L \\ y_L \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & l_s \end{bmatrix} \begin{bmatrix} v_y \\ r \end{bmatrix} + \begin{bmatrix} -v_x \\ -l_s v_x \end{bmatrix} \rho \tag{35}$$

The vision system's parameters are described in Appendix A.

$$\begin{bmatrix} (\alpha_i - 1)W_i & * & * & * & * \\ 0 & -D_i^T D_i & * & * & * \\ 0 & 0 & -F_i^T F_i & * & * \\ 0 & 0 & 0 & 0 & * \\ (W_i - Y_i C_i)P_i A_i & (W_i - Y_i C_i)P_i D_i & (W_i - Y_i C_i)M_i F_i & Y_i F_i & -W_i \end{bmatrix} \prec 0, \forall i \in \mathcal{I} \tag{24}$$

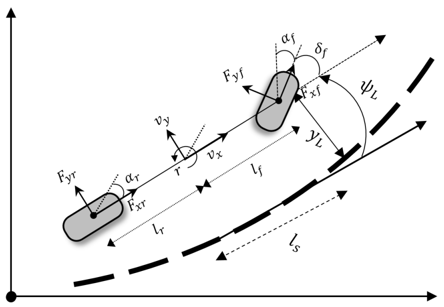

Fig. 1. Bicycle model with vision system measurement.

4.2 Problem formulation

Combining the vehicle lateral dynamics (33) and the vision system model (34) leads to a single dynamical system subject to the road curvature as an unknown input and describing as follows:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t), \;\; y(t) = Cx(t) \tag{36}$$

with the state vector $x = [\beta \; r \; \psi_L \; y_L]^T$, the control input $u(t) = \delta_f$, the unknown input $d(t) = \rho$ and the matrices A, B, and C defined as follows:

$$A = \begin{bmatrix} -\dfrac{c_f + c_r}{m v_x} & \dfrac{c_r l_r - c_f l_f}{m v_x^2} - 1 & 0 & 0 \\ \dfrac{c_r l_r - c_f l_f}{I_z} & -\dfrac{c_r l_r^2 + c_f l_f^2}{I_z v_x} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l_s & v_x & 0 \end{bmatrix}, B = \begin{bmatrix} \dfrac{c_f}{m v_x} \\ \dfrac{c_f l_f}{I_z} \\ 0 \\ 0 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 \\ 0 \\ -v_x \\ -l_s v_x \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that the model (36) describing the vehicle lateral dynamics is subject to longitudinal velocity variations. Thus, a switched representation of the vehicle model is considered where v_x is assumed to be piecewise constant. Then, the model (36) is transformed into a switched discrete-time model [1] :

$$x_{k+1} = A_{\sigma(k)} x_k + B_{\sigma(k)} u_k + E_{\sigma(k)} d_k, \;\; y_k = C x_k \tag{37}$$

where $\sigma(k)$ is the switching signal depending on the measured longitudinal velocity. In order to achieve a more realistic vehicle behaviour, a state disturbance signal $\omega_{\sigma(k)}$ and a measurement noise $v_{\sigma(k)}$ are added to the state and output equations in (37) leading to an uncertain discrete-time switched system of the form (2).

4.3 Experimental data and validation

In this subsection, experimental data are used to evaluate the proposed set-membership state estimator. The measurements are acquired using a prototype vehicle. The test track (located in the city of Versailles-Satory, France) is 3.5km length with various curve profiles allowing vehicle dynamics excitation. Several sensors are implemented on the vehicle: An inertial unit to measure the yaw rate r, an absolute optical encoder to measure the steering angle δ_f and an odometer to provide the vehicle longitudinal speed v_x and a vision system to measure the vehicle angular and lateral displacements ψ_L and y_L as well as the road curvature ρ. For the simulation scenario, three subsystems are defined for $v_x^1 = 3.1m/s$, $v_x^2 = 8.5m/s$ and $v_x^3 = 13.75m/s$ and the following switching law is considered:

$$\sigma(t) = \begin{cases} 1 & if \;\; v_x \in [V_x^0, V_x^1[\\ 2 & if \;\; v_x \in [V_x^1, V_x^2[\\ 3 & if \;\; v_x \in [V_x^2, V_x^3] \end{cases} \tag{38}$$

with $v_x^k = \frac{V_x^k - V_x^{k-1}}{2}$ for $k = 1, 2, 3$. The steering angle, longitudinal velocity, road curvature and the considered switching law are shown in Figure 2. The disturbance and noise vectors satisfy $|\omega_k| \leq [0.004 \; 0.001 \; 0.001 \; 0.001]^T$ and $|v_k| \leq [0.002 \; 0.01 \; 0.002]^T$. The switched matrices $M_{\sigma(k)}$ and $P_{\sigma(k)}$ are obtained by solving constraints (4) using the generalized inverse. The optimal correction switched matrix $\Lambda_{\sigma(k)}$ is designed by solving the optimization problem in Theorem 2, with $\gamma_i = 0.3333$ and $\alpha_i = 0.8$, $\forall i \in \mathcal{I} = \{1, 2, 3\}$. ε_i is obtained using Equation (18) for $s_1 = [1 \; 1 \; 1 \; 1]^T$ and $s_2 [\; 1 \; 1 \; 1]^T$. Detailed expressions of numerical solutions are omitted because of space limitations.

The results of interval estimation are depicted in Fig. 3 which shows that the proposed method allows to obtain a very accurate interval estimation of vehicle state variables.

5. CONCLUSION

In this paper, a set-membership state estimation method based on zonotopes was proposed for uncertain switched discrete-time systems. The proposed approach is suitable for not only linear uncertain switched system but also for switched LPV system that can be reformulated as systems with additive uncertainties. All admissible state trajectories found by the set-membership approach are consistent with both measurements and system model relations. The size of intersection zonotope is minimized at each sample instant by solving a convex LMI-optimization problem. Application to vehicle state estimation demonstrates the efficiency of the proposed approach.

REFERENCES

Alamo, T., Bravo, J.M., and Camacho, E.F. (2005). Guaranteed state estimation by zonotopes. *Automatica*, 41(6), 1035–1043.

Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V. (1994). *Linear matrix inequalities in system and control theory*, volume 15. Siam.

[1] The zero order hold method is used to obtain (37).

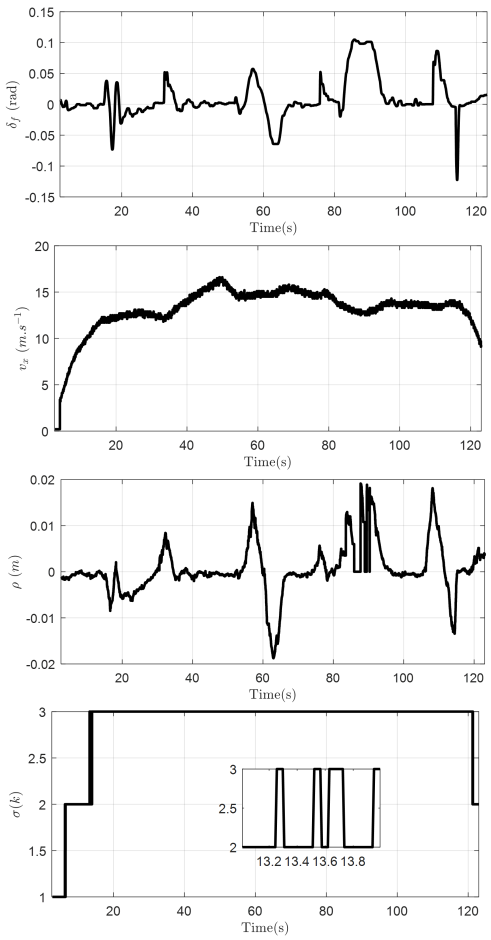

Fig. 2. Steering angle δ_f, Longitudinal velocity v_x, Road curvature ρ and Switching signal $\sigma(k)$.

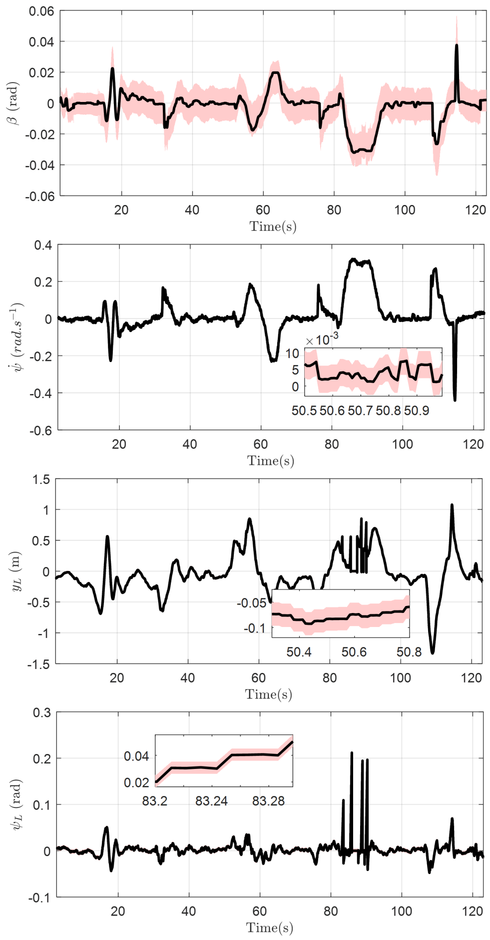

Fig. 3. Interval Estimation of vehicle side slip angle β, yaw rate r, angular and lateral displacements ψ_L and y_L.

Chisci, L., Garulli, A., Vicino, A., and Zappa, G. (1998). Block recursive parallelotopic bounding in set membership identification. *Automatica*, 34(1), 15–22.

Combastel, C. (2015). Zonotopes and kalman observers: Gain optimality under distinct uncertainty paradigms and robust convergence. *Automatica*, 55, 265–273.

Ifqir, S., Ichalal, D., Oufroukh, N.A., and Mammar, S. (2018). Robust interval observer for switched systems with unknown inputs: application to vehicle dynamics estimation. *European Journal of Control*, 44, 3–14.

Le, V.T.H., Alamo, T., Camacho, E.F., Stoica, C., and Dumur, D. (2012). Zonotopic set-membership estimation for interval dynamic systems. In *American Control Conference (ACC), 2012*, 6787–6792. IEEE.

Le, V.T.H., Stoica, C., Alamo, T., Camacho, E.F., and Dumur, D. (2013). *Zonotopes: From guaranteed state-estimation to control*. John Wiley & Sons.

Pacejka, H.B. and Bakker, E. (1992). The magic formula tyre model. *Vehicle system dynamics*, 21(S1), 1–18.

Rajamani, R. (2011). *Vehicle dynamics and control*. Springer Science & Business Media.

Schweppe, F. (1968). Recursive state estimation: Unknown but bounded errors and system inputs. *IEEE Transactions on Automatic Control*, 13(1), 22–28.

Wang, Y., Puig, V., Cembrano, G., and Alamo, T. (2016). Guaranteed state estimation and fault detection based on zonotopes for differential-algebraic-equation systems. In *2016 3rd Conference on Control and Fault-Tolerant Systems (SysTol)*, 478–484. IEEE.

Wang, Y., Wang, Z., Puig, V., and Cembrano, G. (2017). Zonotopic fault estimation filter design for discrete-time descriptor systems. *IFAC-PapersOnLine*, 50(1), 5055–5060.

Appendix A. VEHICLE DYNAMICS AND VISION SYSTEM PARAMETERS

	Lateral Dynamics
F_{yf}, F_{yr}	Lateral tire force of front and rear tires
v_y, v_x	Lateral and longitudinal velocities $(m.s^{-1})$
r	Yaw rate $(rad.s^{-1})$
δ_f	Front steering angle (rad)
c_f, c_r	Cornering stiffness of front and rear tires $(N.rad^{-1})$
l_f, l_r	Distances from front and rear axle to the CG (m)
m	Vehicle mass (kg)
I_z	Moment of inertia $(kg.m^2)$
	Vision System
y_L	Offset displacement at a look ahead distance (m)
ψ_L	Angular displacement at a look ahead distance (rad)
l_s	Look ahead distance (m)
ρ	Road curvature (m^{-1})
β	Sideslip angle (rad)

Measured vehicle specifications for the simulation and experimental validation can be found in Ifqir et al. [2018].