

# A Data-Driven Predictive Control Structure in the Behavioral Framework<sup>\*</sup>

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**Abstract:** This paper presents a data-driven predictive control (DPC) algorithm for linear time-invariant (LTI) systems in the behavioral framework. The system is described by the parametrization of the Hankel matrix constructed from its measured trajectories. The proposed structure follows a two-step procedure. The existence of a controlled behavior is firstly verified from the perspective of dissipativity with the aid of quadratic difference forms (QdFs), then the controlled trajectory is selected from the original uncontrolled behavior through optimization. An illustrative example is presented to demonstrate the effectiveness of the proposed approach.

*Keywords:* data-driven predictive control, behavioral systems theory, dissipativity

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## 1. INTRODUCTION

Due to the rapid processing ability and huge storage capacity of the state-of-the-art technology, the monitoring and control tend to be increasingly relying on the collected data. Not only do data sets describe the dynamical features of a system more accurately than models, but the rich information contained within the data sets can provide valuable insights to the dynamics of the systems. One of the classical way is to find a regression model that best describes the data set, but such a strategy defeats the purpose because numerous assumptions, hence inaccuracies, are introduced in the process of modelling. If data-driven strategies were to be developed, then the function of the data set is to describe the system instead of being merely an aid for the search of a model. Huang and Kadali (2008) developed a subspace approach to data-driven control for LTI systems.

Proposed by Willems (1991), the behavioral systems theory views a dynamical system as a collection of functions, or trajectories, called the behavior, that map a time axis to a signal space. The theory claims that what defines a dynamical system is its trajectories rather than its representations (Polderman and Willems, 1998). This theory unites another theory developed by Willems as well, the dissipativity theory (Willems, 1972a), into this framework as well: dissipativity is simply a viewpoint to represent the dynamical features of a system. It is a stand-alone representation rather than a property (Willems and Takaba, 2007). This view is the exact same rationale as data-driven approaches and has hence drawn attentions in the past few years. Willems et al. (2005) showed that a Hankel matrix constructed by a measured trajectory from an LTI system can parametrize all of the trajectories in the behavioral set, provided that the input is sufficiently excited. This

is similar to the subspace identification approach (Huang and Kadali, 2008) but from a different perspective.

With the shift from model-based to data-driven control, the basic viewpoint for control should also change. The desired trajectory is not created, it is selected from the original system. In other words, control is possible if and only if the desired controlled trajectory is *already* within the system, and the purpose of the controller is to pose restriction on the system so that only the desired trajectories can happen (Polderman and Willems, 1998). This in theory means that control can be carried out in a very simple way if the controlled behavior can be verified to be contained in the original system. Markovskiy and Rapisarda (2008) proposed several algorithms for data-driven control with this rationale and they were perfected in Maupong and Rapisarda (2017). A data-driven predictive control (DPC) algorithm was also formulated in Coulson et al. (2019) and Berberich et al. (2019). However, the algorithms all consider the case where the system is noise-free, which in reality never happens. With the presence of noise, the actual trajectory *no longer* belongs to the behavioral set described by the aforementioned method. To the best of the authors knowledge, little has been done in DPC formulation using the behavioral framework with the consideration of noise. The existence of a controlled behavior was discussed in details in Willems and Trentelman (2002) in a model-based setting but is otherwise scarcely discussed about.

The goal of this paper is to develop a DPC structure using the behavioral approach. As illustrated above, control in the behavioral framework involves the verification of the existence of the controlled behavior and the actual control implementation. We use the viewpoint of dissipative dynamical system to provide a set of sufficient conditions for the existence of the controlled behavior, then we use Hankel matrix, together with dissipativity description of the desired behavior, to formulate the DPC optimization problem. Note that the concept of data-driven dissipativity has already been discussed in Maupong et al. (2017)

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and Romer et al. (2019). But the recursive verification of dissipativity was not discussed.

The structure of the rest of the paper is as follows. In Section 2, background information of behavior theory and dissipative system theory is presented. Section 3 provides sufficient conditions for the existence of the controlled behavior as well as the DPC structure. An illustrative example is presented in Section 4. Section 5 concludes the paper.

**Notation.** We use the conventional notations  $\mathbb{R}$ ,  $\mathbb{Z}$ ,  $\mathbb{R}^n$ ,  $\mathbb{R}^{m \times n}$  and  $\mathbb{R}^{m \times n}[\cdot]$  to denote, respectively, the set of all real numbers, integers,  $n$ -dimensional real vectors  $m \times n$  real matrices and  $m \times n$  polynomial matrices with real coefficients. We use  $\mathbb{R}^\bullet$ ,  $\mathbb{R}^{\bullet \times n}$ , etc., to denote the space of vectors and matrices with unknown but finite dimensions. We use  $\mathbb{S}^\bullet[\cdot]$  to denote the set of symmetric polynomial matrices. We denote the set of all integers no greater than  $L$  as  $\mathbb{Z}_L^+$ . The generic variable of a space  $\mathbb{W}$  is denoted as  $w$  with dimension denoted as  $w$ . We denote the set of all mappings from a space  $\mathbb{T}$  to another space  $\mathbb{W}$  as  $\mathbb{W}^\mathbb{T}$ . An identity matrix with dimension  $n \times n$  is denoted by  $I_n$ .

## 2. PRELIMINARIES

### 2.1 Behavioral Systems Theory

In the behavioral framework, a dynamical system  $\Sigma$  is defined as a triple  $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ , where  $\mathbb{T}$  represents the time axis,  $\mathbb{W}$  is the signal space of the system and  $\mathfrak{B} \subset \mathbb{W}^\mathbb{T}$  is the behavior of the system (Willems, 1991). The generic variable  $w$  of the space  $\mathbb{W}$  is called the *manifest* variable. If  $\mathbb{W}$  is a vector space,  $\mathfrak{B}$  is a linear subspace of  $\mathbb{W}^\mathbb{T}$  and  $\sigma\mathfrak{B} \subset \mathfrak{B}$ , where  $\sigma$  is a discrete-time shift operator, then  $\Sigma$  is linear and time-invariant (Polderman and Willems, 1998). In many cases, the description of a behavior relies on the aid of auxiliary variables called the *latent* variable denoted by  $\ell$ . A dynamical system with latent variable can then be defined as a quadruple  $\Sigma_{full} = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{full})$  where  $\mathfrak{B}_{full} \subset (\mathbb{W} \times \mathbb{L})^\mathbb{T}$  is the *full* behavior. The manifest behavior is then  $\mathfrak{B} = \{w \mid \exists \ell, (w, \ell) \in \mathfrak{B}_{full}\}$ . Since the main focus of this paper is date-driven control, we assume throughout this paper that the time axis is the set of all positive integers, i.e.,  $\mathbb{T} = \mathbb{Z}^+$ . A truncated trajectory from  $w$  on the interval  $[1, L]$  is denoted at  $w|_{[1, L]}$ . The behavior restricted to the interval  $[1, L]$  can then be defined as

$$\mathfrak{B}|_{[1, L]} = \left\{ w \in (\mathbb{R}^w)^{\mathbb{Z}_L^+} \mid \exists w' \in \mathfrak{B}, w = w'|_{[1, L]} \right\}.$$

While  $\mathfrak{B}$  admits multiple representations, the most general representation is the *latent variable* representation

$$R(\sigma)w = M(\sigma)\ell \quad (1)$$

where

$$R(\sigma) = \sum_{i=0}^{N_R} R_i \sigma^i \in \mathbb{R}^{\bullet \times w}[\sigma], \quad M(\sigma) = \sum_{i=0}^{N_M} M_i \sigma^i \in \mathbb{R}^{\bullet \times 1}[\sigma]$$

are the coefficient matrices. If the system is controllable, i.e., it is always possible to move from one trajectory to any other trajectories in the behavior within finite time, then it is possible to have  $R(\sigma) = I_w$ , reducing (1) to the *image* representation

$$w = M(\sigma)\ell. \quad (2)$$

If a part of the variables in  $w$ , call it  $u$ , is such that for any trajectory of  $u$ , there exists a trajectory of the remaining part of  $w$ , call it  $y$ , such that  $(y, u) \in \mathfrak{B}$ , then  $u$  is said to be free. If all variables in  $u$  are free while none of that in  $y$  is, then  $w = (y, u)$  is called an *input/output* partition of  $w$ .

In data-driven control, obviously the model of the system is not available. However, for LTI systems, one appropriately chosen trajectory is enough to parametrize the entire behavior restricted to a certain interval. For a given measured trajectory  $\tilde{w} \in \mathfrak{B}|_{[1, T]}$ , it is possible to construct a Hankel matrix of order  $L$  as

$$\mathfrak{H}_L(\tilde{w}) = \begin{bmatrix} \tilde{w}(1) & \tilde{w}(2) & \cdots & \tilde{w}(T-L+1) \\ \tilde{w}(2) & \tilde{w}(3) & \cdots & \tilde{w}(T-L+2) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{w}(L) & \tilde{w}(L+1) & \cdots & \tilde{w}(T) \end{bmatrix}. \quad (3)$$

We also define a portion of the Hankel matrix constructed from a trajectory  $\tilde{w}$  with depth  $l$  starting from the  $k$ th block row ( $k \neq 1$ ) as  $\mathfrak{H}_{l, k}(\tilde{w})$ . A signal  $w$  is called persistently exciting of order  $L$  if  $\text{rank}(\mathfrak{H}_L(\tilde{w})) = Lw$  (Willems et al., 2005; Huang and Kadali, 2008). The behavior  $\mathfrak{B}|_{[1, L]}$  can be parametrized by the Hankel matrix according to the following lemma.

*Lemma 1.* (Willems et al. (2005)). Assume  $\mathfrak{B}$  is controllable and  $(y, u)$ , where  $u$  is free, is an input/output partition of  $w$ . Let  $\tilde{w} \in \mathfrak{B}|_{[1, T]}$ . If  $\tilde{w}$  is persistently exciting of order  $L + \mathfrak{n}(\mathfrak{B})$ , where  $\mathfrak{n}(\mathfrak{B})$  is the McMillan degree of  $\mathfrak{B}$ , then  $\text{colspan}(\mathfrak{H}_L(\tilde{w})) = \mathfrak{B}|_{[1, L]}$ . In other words, there exists  $g \in \mathbb{R}^{T-L+1}$  such that

$$\tilde{v} = \mathfrak{H}_L(\tilde{w})g \quad (4)$$

for all  $\tilde{v} \in \mathfrak{B}|_{[1, L]}$ .

Notice that the structure of (4) is similar to that of an image representation.

### 2.2 Dissipative Systems Theory

Dissipativity is introduced by Willems (1972a,b) to analyse the dynamics of a system as well as interconnections. While initially defined in state space, we introduce the notion of dissipativity using the manifest variables because the focus is on the input/output relationships. A dynamical system  $\Sigma$  is dissipative if there exists a positive semi definite storage  $V(w(k))$  and supply rate  $S(w(k))$  satisfying (Yan et al., 2019a)

$$V(w(T)) - V(w(0)) \leq \sum_{k=0}^T S(w(k)) \quad (5)$$

for all integers  $T \geq 0$ . If (5) holds for all  $T \in [1, L] \cap \mathbb{Z}$ , then  $\Sigma$  is said to be  $L$ -dissipative (Romer et al., 2019).

While quadratic supply rates are sufficient to capture the dynamical features of an LTI system in state space, more elaborate description is needed for a system defined on the external signal space. Therefore, quadratic difference forms (QdF), which is defined on the *extended* signal space, has been proposed (Kojima and Takaba, 2005) and is defined as

$$Q_\Phi(w(k)) = \sum_{p=0}^K \sum_{q=0}^K w(k+p)^T \Phi_{pq} w(k+q). \quad (6)$$

where  $K$  is called the order of QdF. It is said to be induced by a two-variable polynomial matrix  $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$ , where  $\zeta$  and  $\eta$  are the shift operators for  $w^T$  and  $w$ , respectively, and

$$\Phi(\zeta, \eta) = \sum_{p=0}^K \sum_{q=0}^K \zeta^p \Phi_{pq} \eta^q. \quad (7)$$

The coefficient matrix of  $\Phi(\zeta, \eta)$  is defined as

$$\tilde{\Phi} = \begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \Phi_{0K} \\ \Phi_{10} & \Phi_{11} & \cdots & \Phi_{1K} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{K0} & \Phi_{K1} & \cdots & \Phi_{KK} \end{bmatrix}.$$

$\Phi(\zeta, \eta)$  is positive (respectively, non-negative), denoted by  $\Phi \succ 0$  (respectively,  $\Phi \succeq 0$ ) if and only if its coefficient matrix  $\tilde{\Phi}$  is positive definite (respectively, semi-definite). The dual operator  $\star$  is defined as  $\Phi^\star(\zeta, \eta) = \Phi^T(\eta, \zeta)$ .  $\Phi \in \mathbb{S}^w[\zeta, \eta]$  if and only if  $\Phi = \Phi^\star$  if and only if  $\tilde{\Phi} = \tilde{\Phi}^T$ . The rate of change of QdF is denoted as

$$Q_{\nabla\Phi}(w(k)) = Q_\Phi(w(k+1)) - Q_\Phi(w(k)), \quad (8)$$

and is induced by

$$\nabla\Phi(\zeta, \eta) = (\zeta\eta - 1)\Phi(\zeta, \eta). \quad (9)$$

*Definition 1.* ( $\Phi$ -dissipativity (Kaneko and Fujii, 2000)). A dynamical system  $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$  is dissipative with respect to a QdF  $Q_\Phi(w(k))$  if there exists a storage function  $Q_\Psi(w(k))$  such that

$$Q_\Psi(w(k)) \geq 0 \quad (10a)$$

$$Q_\Psi(w(k+1)) - Q_\Psi(w(k)) \leq Q_\Phi(w(k)) \quad (10b)$$

hold for all  $w \in \mathfrak{B}$ .

The notion of  $\Phi$ - $L$ -dissipativity can then be defined analogously (Maupong et al., 2017).

### 3. DATA-DRIVEN PREDICTIVE CONTROL

Many control objectives such as trajectory tracking and disturbance attenuation can be represented as a dissipativity property of controlled behavior with respect to a certain supply rate. In this section we introduce the structure of data-driven predictive control (DPC) that renders an LTI system represented by a Hankel matrix constructed from one of its measured trajectories dissipative with respect to a QdF induced by  $\Phi_d(\zeta, \eta)$ . The rationale for control design in the behavioral framework is slightly different than that in the conventional sense: rather than creating a stable closed-loop system, the controller is simply selecting the desirable trajectories from the system and restrict the outcome to only these trajectories through interconnection. Therefore, the difficulty of control design lies in the verification of the existence of a controlled behavior rather than designing the controller itself. We begin by giving a sufficient condition for the existence of the controlled behavior. The trajectory from the desired behavior is then directly selected from the system using the desired dissipativity condition.

#### 3.1 Existence of Desired Controlled Behavior

While the process of interconnecting difference systems is rather cumbersome, interconnecting dissipative dynamical

systems is much less demanding, in that the supply rate of the interconnected system is simply the linear combination of that of the subsystems (Willems, 1972a). We therefore provide conditions for the existence of controlled behavior from the perspective of dissipativity. Without loss of generality, we assume in this paper that the order of the storage function is one less than that of the supply rate, i.e.,  $K_\Phi = K_\Psi + 1 = K_{\nabla\Phi} = K$ . Note that it is always possible to augment either  $\Phi$  or  $\Psi$  to meet this requirement by adding zeros to the polynomial matrix.

*Theorem 1.* Given a controllable dynamical system

$$\Sigma = \left( \mathbb{Z}_L^+, \mathbb{R}^w, \text{colspan}(\mathfrak{H}_L(\tilde{w})) \right) \quad (11)$$

whose behavior is constructed according to Lemma 1 and a QdF induced by  $\Phi \in \mathbb{S}^w[\zeta, \eta]$ , the following statements are equivalent:

(1)  $\Sigma$  is  $\Phi$ - $(L-K)$ -dissipative;

(2)  $\sum_{k=1}^{L-K} \mathfrak{H}_{K+1,k}^T(\tilde{w}) \tilde{\Phi} \mathfrak{H}_{K+1,k}(\tilde{w}) \geq 0$ ;

(3) There exists a QdF induced by  $\Psi \in \mathbb{S}^w[\zeta, \eta]$  such that

$$\mathfrak{H}_{K,k_1}^T(\tilde{w}) \tilde{\Psi} \mathfrak{H}_{K,k_1}(\tilde{w}) \geq 0 \quad (12a)$$

$$\mathfrak{H}_{K+1,k_2}^T(\tilde{w}) (\tilde{\Phi} - \nabla\tilde{\Psi}) \mathfrak{H}_{K+1,k_2}(\tilde{w}) \geq 0 \quad (12b)$$

for all  $k_1 \in [1, L-K+1] \cap \mathbb{Z}$  and  $k_2 \in [1, L-K] \cap \mathbb{Z}$ .

**Proof.** The equivalence of (1) and (2) have been proven in Romer et al. (2019). Here we only prove the equivalence of (3) and (1). Multiplying both sides of both inequalities in (12) gives (10) for all  $k \in [1, L-K] \cap \mathbb{Z}$ , which leads to the dissipativity condition in (1).  $\square$

*Remark 1.* It is interesting to note that while the Hankel matrix parametrizes the behavior up to  $L$  steps, dissipativity can only be verified up to  $L-K$  steps. This is due to the fact that QdFs include future steps to obtain more detailed information about the dynamics of the system. Each  $K+1$  steps can only verify dissipativity for *one* step, meaning that dissipativity verification for the  $(L-K)$ th step has already used up all steps in the Hankel matrix.

*Remark 2.* Notice that (12a) actually leads to the non-negativity of  $Q_\Psi(w(k))$  up to  $L-K+1$  steps. The reason for this is that although the  $(L-K+1)$ th step is not required for the storage function, it is used in (12b) and therefore must still be a valid storage function for this step.

Assuming that the controller  $\Sigma_c = \left( \mathbb{Z}_L^+, \mathbb{R}^c, \mathfrak{B}_c \right)$  has manifest variable that admits an input/output partition  $c = (y_c, u_c)$ , then it is possible to associate it with a supply rate

$$Q_{\Phi_c}(c(k)) = c^T \Phi_c c = \begin{bmatrix} y_c \\ u_c \end{bmatrix}^T \begin{bmatrix} Q_c(\zeta, \eta) & \mathcal{S}_c(\zeta, \eta) \\ \mathcal{S}_c^\star(\zeta, \eta) & \mathcal{R}_c(\zeta, \eta) \end{bmatrix} \begin{bmatrix} y_c \\ u_c \end{bmatrix} \quad (13)$$

where  $Q_c \in \mathbb{S}^{y_c}[\zeta, \eta]$ ,  $\mathcal{S}_c \in \mathbb{R}^{y_c \times u_c}[\zeta, \eta]$  and  $\mathcal{R}_c \in \mathbb{S}^{u_c}[\zeta, \eta]$ . The supply rate for the controlled system,  $Q_{\Phi_{cd}}(w_d(k))$ , can then be represented by

$$Q_{\Phi_{cd}}(w_d(k)) = Q_\Phi(w(k)) + Q_{\Phi_c}(c(k)) = w_d^T \Phi_{cd}(\zeta, \eta) w_d \quad (14)$$

where  $w_d$  column vector containing all system and controller manifest variables,  $w = \Pi_p w_d$ ,  $c = \Pi_c w_d$ , and

$$\Phi_{cd} = \Pi_p^T \Phi \Pi_p + \Pi_c^T \Phi_c \Pi_c. \quad (15)$$

For a chosen controlled behavior, the existence of a controlled behavior for a dynamical system can then be verified according to the following proposition.

*Proposition 2.* For a dynamical system (11) and a QdF induced by  $\Phi_d \in \mathbb{S}^{w_a}[\zeta, \eta]$ , there exists a controlled behavior that is  $\Phi_d$ - $(L - K)$ -dissipative if there exist  $\Phi \in \mathbb{S}^w[\zeta, \eta]$ ,  $\Psi \in \mathbb{S}^w[\zeta, \eta]$  and  $\Phi_c \in \mathbb{S}^c[\zeta, \eta]$  such that (12) is satisfied for all  $k_1 \in [1, L - K + 1] \cap \mathbb{Z}$  and  $k_2 \in [1, L - K] \cap \mathbb{Z}$ , and

$$\Phi_{cd} - \Phi_d \leq 0, \quad (16a)$$

$$\mathcal{Q}_c < 0, \mathcal{R}_c > 0 \text{ or } \mathcal{Q}_c > 0, \mathcal{R}_c < 0. \quad (16b)$$

**Proof.** The combination of (12) and (16a) ensure the existence of a controlled behavior. However, since there is no lower bound for  $\Phi_c$ , (16a) can always be satisfied if  $\Phi_c$  is negative with arbitrarily large magnitude. Since there is no non-trivial trajectory that is dissipative with respect this supply rate, it must be excluded, and (16b) is one way to guarantee the avoidance of such supply rates. Therefore, if all conditions are satisfied, there exists a desired controlled behavior that can be implemented by the controller.  $\square$

*Remark 3.* A part of conditions in Proposition 2 have been presented in Wang et al. (2019), Tippett and Bao (2013) and Yan et al. (2019b). In this paper we formulate the conditions entirely based on data as opposed to using given models or identifying a set of models in state space. Furthermore, rather than assuming the controller variable as a subset of the system variable, we consider the case where neither  $w$  nor  $c$  is a subset of the other, thereby encompassing a wider range of problems.

### 3.2 The DPC Control Structure

Suppose that a dynamical system  $\Sigma$  is described by (4). With suitable permutations and partitions, (4) can be rewritten as

$$\begin{bmatrix} \tilde{w}_p \\ \tilde{w}_f \end{bmatrix} = \begin{bmatrix} \mathfrak{H}_M(\tilde{w}) \\ \mathfrak{H}_{L-M, M+1}(\tilde{w}) \end{bmatrix} g \quad (17)$$

where  $w_p$  and  $w_f$  are, respectively, the past and future trajectory, and  $M$  is an integer. Note that  $M$  needs to be larger than the lag of the system  $L(\mathfrak{B})$  (i.e., the smallest number of steps for the past and the future to be independent) because only with at least  $L(\mathfrak{B})$  step overlap for each iteration can we form a trajectory that is from  $\mathfrak{B}_{[1, L]}$  (Markovsky et al., 2005). Furthermore, denoting the component in  $c$  that is not shared with  $w$  as  $w^f$  (because they are essentially “free”), the future trajectory for the manifest variable of the controlled system can be represented as

$$\begin{aligned} \tilde{w}_{d[M+1, L]} &= \Pi \begin{bmatrix} \tilde{w}_f \\ \tilde{w}_f \end{bmatrix} \\ &= \Pi \begin{bmatrix} \mathfrak{H}_{L-M, M+1}(\tilde{w}) & 0 \\ 0 & I_{(M+1)w^f} \end{bmatrix} \begin{bmatrix} g \\ \tilde{w}_f^f \end{bmatrix} \\ &:= \mathfrak{H}_{df} \begin{bmatrix} g \\ \tilde{w}_f^f \end{bmatrix}. \end{aligned} \quad (18)$$

where  $\tilde{w}_f^f$  is the future trajectory of the free variable and  $\Pi$  is a permutation matrix.

Let the order of  $\Phi_d$  be  $K_d$ . Since for each iteration,  $M$  steps are used as the past trajectory, then according to

Theorem 1, the number of future steps that can be verified to be dissipative is at most  $L_f = L - K_d - M$ . We therefore have

$$\sum_{k=1}^{L_f} \tilde{w}_{d[M+k, M+K_d+k]}^T \tilde{\Phi}_d \tilde{w}_{d[M+k, M+K_d+k]} \quad (19)$$

$$= \begin{bmatrix} \tilde{w}_{d[M+1, M+K_d+1]} \\ \vdots \\ \tilde{w}_{d[L-K_d, L]} \end{bmatrix}^T \left( I_{L_f} \otimes \tilde{\Phi}_d \right) \begin{bmatrix} \tilde{w}_{d[M+1, M+K_d+1]} \\ \vdots \\ \tilde{w}_{d[L-K_d, L]} \end{bmatrix} \quad (20)$$

$$= \tilde{w}_{d[M+1, L]}^T P^T \left( I_{L_f} \otimes \tilde{\Phi}_d \right) P \tilde{w}_{d[M+1, L]} \quad (21)$$

$$= \begin{bmatrix} g \\ \tilde{w}_f^f \end{bmatrix}^T \mathfrak{H}_{df}^T P^T \left( I_{L_f} \otimes \tilde{\Phi}_d \right) P \mathfrak{H}_{df} \begin{bmatrix} g \\ \tilde{w}_f^f \end{bmatrix} \quad (22)$$

where  $\otimes$  is the Kronecker product and  $P$  is a permutation matrix, i.e.,

$$\begin{bmatrix} \tilde{w}_{d[M+1, M+K_d+1]} \\ \vdots \\ \tilde{w}_{d[L-K_d, L]} \end{bmatrix} = P \tilde{w}_{d[M+1, L]}. \quad (23)$$

Assuming that the Hankel matrix is constructed from a noise-free trajectory but input and measurement noises are present during implementation, the DPC problem can be stated as

$$\begin{aligned} \min_g \quad & \sum_{k=1}^{L_f} C(w_{df}(k)) + \varpi \varepsilon^2 \\ \text{s.t.} \quad & \begin{bmatrix} g \\ \tilde{w}_f^f \end{bmatrix}^T \mathfrak{H}_{df}^T P^T \left( I_{L_f} \otimes \tilde{\Phi}_d \right) P \mathfrak{H}_{df} \begin{bmatrix} g \\ \tilde{w}_f^f \end{bmatrix} \geq 0 \end{aligned} \quad (24)$$

where  $w_{df}$  is the future trajectory of the manifest variable of the controlled system,  $C(\cdot)$  is a to-be-specified cost (e.g., reference tracking, economic cost, etc.) and  $\varpi$  is a weighting. The estimation error  $\varepsilon$  between past estimation using Hankel matrix the actual past history trajectory can be calculated as

$$\varepsilon = \|\mathfrak{H}_M(\tilde{w})g - \tilde{w}_p\|. \quad (25)$$

The future trajectory can then be computed as

$$\tilde{w}_f = \mathfrak{H}_{L-M, M+1}(\tilde{w})\hat{g}. \quad (26)$$

Note that  $\varepsilon = 0$  has been used as a constraint to compute  $g$  in Markovsky and Rapisarda (2008) and Berberich et al. (2019). Such a strategy cannot be applied to the current situation because it considers the actual trajectories during implementation to be noise-free. A noisy trajectory is very close but not within the behavior  $\mathfrak{B}$  and hence can only be *approximately* parametrized by the Hankel matrix. By introducing  $\epsilon$  and the weighting into the objective, a trade-off can be carried out between the desired performance and the accuracy of fitting into the historical data to search for the best possible outcome with the presence of noise. In each iteration, a vector  $g$  is computed through optimization and the predicted future trajectory can subsequently be deduced. The introduction of dissipativity condition guarantees that the predicted trajectory is both from the system behavior (as it is parametrized by the Hankel matrix) and is dissipative with respect to the desired supply rate  $\Phi_d$  according to Theorem 1(2). The structure proposed in this section is very general and the

cost  $C(\cdot)$  can be chosen according to the control goal and the situation at hand.

#### 4. ILLUSTRATIVE EXAMPLE

In this section we consider the tracking control of a distillation column. The goal is to regulate the temperature of a tray near the top ( $y_1$ ) and another near the bottom ( $y_2$ ) by manipulating liquid reflux ( $u_1$ ) and vapor boilup ( $u_2$ ). A transfer function at an operating point is as follows:

$$G(s) = \begin{bmatrix} \frac{-33.89}{(98.02s + 1)(0.42s + 1)} & \frac{32.63}{(99.6s + 1)(0.35s + 1)} \\ \frac{-18.85}{(75.43s + 1)(0.30s + 1)} & \frac{34.84}{(110.5s + 1)(0.03s + 1)} \end{bmatrix} \quad (27)$$

The system is discretized with a sampling rate of 1 minute. A trajectory of 100 steps is generated using random bang-bang control to ensure persistent excitation. The trajectory of the first input is shown in Figure 1. A Hankel matrix with depth  $L = 12$  is then formulated and partitioned with  $M = 9$ , which is much higher than  $L(\mathfrak{B})$ .

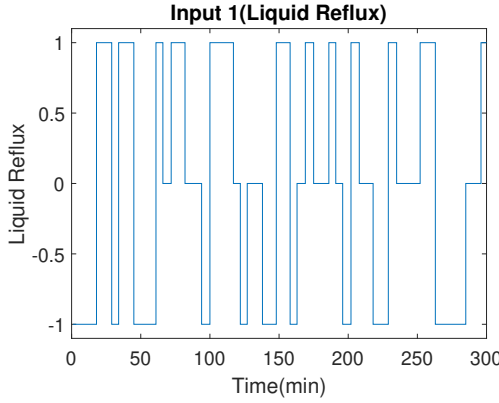


Fig. 1. Random Bang-bang Control Input 1

In this case  $w = \text{col}(y, u)$ ,  $c = (y, u, r)$  where  $r$  is the reference trajectory. The desired controlled behavior is chosen as the set of trajectories satisfying the weighted  $\mathcal{H}_\infty$  condition  $\|WT_{er}\|_\infty \leq 1$  where  $W(z) = \frac{N(z)}{d(z)}$  is a weighting function and  $T_{er}(z)$  is the transfer function from reference to tracking error. After interconnection, the manifest variable is then  $w_d = \text{col}(e, u, r)$  where  $e = r - y$  is the tracking error. The weighting function is then chosen to be

$$W(z) = \frac{z - 0.5}{z - 0.95} I_2 \quad (28)$$

and therefore

$$Q_{\Phi_d}(w_d) = \begin{bmatrix} e \\ u \\ r \end{bmatrix}^T \begin{bmatrix} -n(\zeta)n(\eta)I_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d(\zeta)d(\eta)I_2 \end{bmatrix} \begin{bmatrix} e \\ u \\ r \end{bmatrix} \quad (29)$$

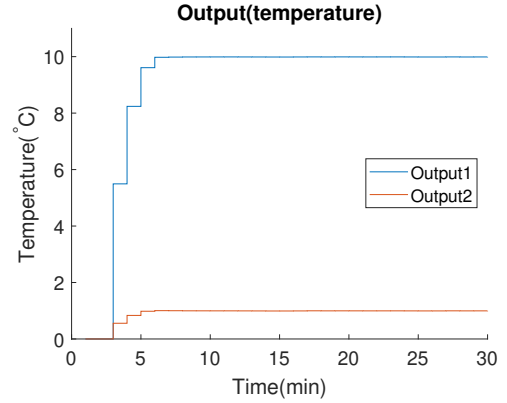
where  $n(\sigma) = \sigma - 0.5$  and  $d(\sigma) = \sigma - 0.95$ . Obviously  $w^f = r$ ,  $K_d = 1$  and  $L_f = 2$ . The cost for this case is chosen as

$$C(w_{df}(k)) = 10^4 \|e_f(k)\|^2 + \|u_f(k)\|^2, \quad (30)$$

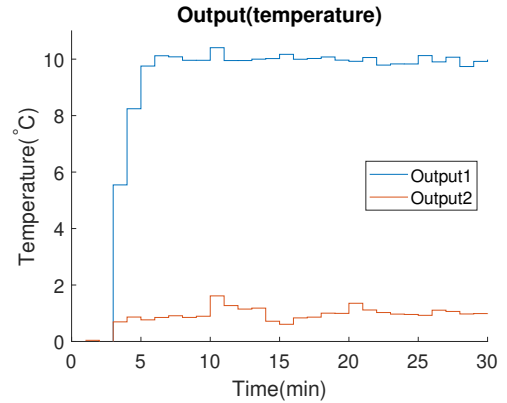
which means that the main focus is on the performance of the predicted trajectory. The existence of controlled

behavior is then checked according to Proposition 2. For this case we choose to use  $\mathcal{Q}_c \succ 0$ ,  $\mathcal{R}_c \prec 0$  in (16b) and  $\Phi_d$  is verified to be achievable through control.

With the references being  $r_1 = 10$  and  $r_2 = 1$ , simulations are carried out for both noise-free case and the noisy case with noises of power -20dB added to both control inputs and output measurements. The control outcomes are shown in Figure 2. As is shown in the output trajectories in Figure 2a and Figure 2b, tracking can be effectively achieved with or without the presence of noise.



(a) Reference tracking using noiseless data



(b) Reference tracking using on noisy data

Fig. 2. Simulation Results

#### 5. CONCLUSION

In this paper, a DPC structure has been formulated to control an LTI system parametrized by a Hankel matrix constructed by its measured trajectories using the behavioral framework. Sufficient conditions for the existence of controlled have been developed in terms of dissipativity. The optimization structure includes a cost function that enables the trade-off between performance and accuracy of data fitting and the controlled trajectory is selected directly from the original behavioral set. A simulation study has been carried out to illustrate the proposed approach under input and measurement noise. Possible future directions include the extension to the control of nonlinear systems and distributed control of large-scale interconnected systems.

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