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Development of a structure identification method for nonlinear SISO systems

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Abstract: The basis of nonlinear system identification underlies a proper functional structure provided through either a detailed physical process description or a-priori knowledge from experts. However, these conditions are not provided in many engineering disciplines due to continuously changing functional structures depending on acting operational points and complex plant operations, which makes classical White-Box- or Grey-Box-Modelling difficult or even impossible. In order to achieve a reliable performance in nonlinear system identification, this paper seeks to examine a data-driven approach to identify the functional structure for the special case of nonlinear single-input-single-output (SISO) systems. The identified functional structures from the proposed method will be embedded as nonlinear candidates into the sparse regression method as system identification procedure and the performance of the estimation will be observed.

Keywords: **Structure identification, nonlinear system identification, Black-Box-Modelling, data-driven analysis**

1. INTRODUCTION

System identification forms a fundamental basis for further investigations in many engineering disciplines. In order to achieve sustainable and reliable results in further analysis, the reliability of the identified model must be ensured.

However, the known linear approaches for system identification (Shardt (2015), Unbehauen et al. (2016), Ljung (1999)) are not applicable to nonlinear cases, since both the transition between time-domain and frequency domain is non-existent and the pre-defined structure identification resulting from the models is not given. Classical nonlinear system identification approaches consist of neural network approaches, block-oriented system identification or sum-series application, in which either the structure is provided arbitrarily through a base function, a static nonlinearity or in case of sum-series without a structure until the fit of the data is promising (Narendra et al. (1990), Nelles. (2001), Unbehauen et al. (2016), Hofmann. (2003)).

Thus, in this paper, based on the previously mentioned challenges, a theoretical derivation of the results for nonlinear, deterministic SISO structure identification is presented that takes into consideration the measurements from an open loop condition. The theoretical result is used to modify the sparse regression method of Brunton et al. (2015) to provide the required nonlinear candidates for a proper reduced model order. The overall performance is then validated using an inverted pendulum as a benchmark system in a simulation environment.

2. THEORY

Consider an open-loop nonlinear SISO system, consisting of the input signal u_t, the output signal y_t and the unknown nonlinear dynamics described in the time domain as f containing the state expressions x_t and the unknown transition between output and states C such that the overall expression will be

$$\dot{x}_t = f(x_t, u)$$
$$y_t = C(x, u) \tag{1}$$

Here, the assumption is made that the sampling is sufficient containing the dynamic information and the output is directly connected to the states. Furthermore, the obtained results are applicable both in time and frequency-domain, since the dynamics acts in both representations. To prove this statement the structure identification is derived in the frequency domain, while the system identification is conducted in time domain in order to determine a state-space representation. Based on these circumstances, the structure identification method will be applied in the following manner as shown in Figure 1.

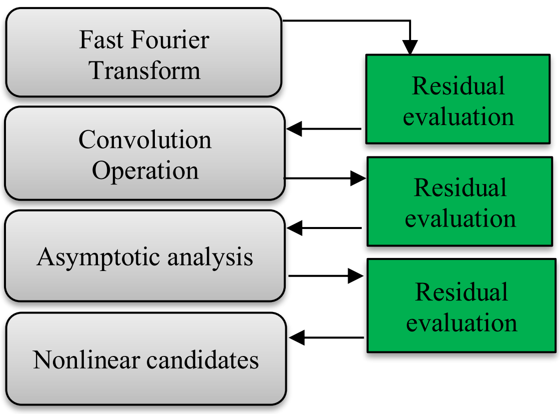

Figure 1: Overview of the structure identification method

The major objective of the proposed structure identification method consists of providing the possible, available nonlinear candidates derived from the dynamic behaviour of the system either from excitation or sufficient measurements. While the Fast Fourier Transform is applied to determine possible harmonic function candidates at the beginning, the convolution operation and the asymptotic analysis is used to cover all other existing nonlinear candidates except for non-bijective functions. In order to ensure that the loop can be broken in case no further dynamics are available, a residual evaluation is implemented to save the computational effort. Hence, the initial output signal can be expressed as

$$y_t = h_t + c_t + a_t + e_t \tag{2}$$

Eq.2 is used as a regression indicator including the determined harmonic part h_t, the convolution part c_t and the asymptotic part a_t with respect to the estimation error e_t.

2.1. Fast Fourier Transform

Fast Fourier Transform will be used to filter out the possible, existing harmonic part h_t of the output signal y_t. Therefore, the harmonic part with N samplings can be expressed as:

$$h_t = \sum_{n=1}^{N} a_n \cos(\omega_n t) + \sum_{j=1}^{N} b_j \sin(\omega_j t) \tag{3}$$

Applying the Fourier Transform results

$$H_f = \sum_{n=1}^{N} \frac{a_n}{2}[\delta(\omega+\omega_n) + \delta(\omega-\omega_n)] + \sum_{j=1}^{N} \frac{b_j}{2} i[\delta(\omega+\omega_j) + \delta(\omega-\omega_j)] \tag{4}$$

where $\delta(.)$ is the Dirac function and i an imaginary number. The result from Eq.4 is complex, wherefore the sinusoidal expression will be replaced by a phase shifted cosine expression. In order to achieve the Fast Fourier Transform of the real Fourier series, the classical Fourier Transform is modified with the Cooley-Tukey Algorithm, which gives

$$H_f = \sum_{n=1}^{2N} \frac{a_n}{2}[\delta(\omega+\omega_n) + \delta(\omega-\omega_n)] \tag{5}$$

and provides the necessary frequencies to cancel out the harmonic part from Eq. 2.

2.2. Convolution operation

Starting from the condition stated in Eq. 5 and assuming the residual evaluation resulted a non-white sequence, the convolution operation is used both to determine possible underlying polynomial or exponential functional structures and to distinguish in between, since in time-domain the possibility of approximating functions with higher order polynomial is given.

2.2.1.Convolution operation for polynomial identification

In general, a polynomial function p_t can be written as

$$p_t = \sum_{i=1}^{n} a_i t^i \tag{6}$$

where i denotes the polynomial order with n samplings and a_i the related weightings. In order to obtain a polynomial library based on the convolution operation, the correspondence of the Laplace Transform for convolution operation is used, that is,

$$\mathcal{L}\{p_t * p_t\} = \mathcal{L}\{\int_0^{\infty} p(u)p(t-u)du\} = \mathcal{L}\{p_t\} \cdot \mathcal{L}\{p_t\} \tag{7}$$

Theorem 1. Using Eq.7, it can be obtained that the convolution of two polynomials itself results into a piecewise higher order polynomial P with $2n$ samples, T_z denoting the intermediate time stamps and T_m describing the last time stamps, that is

$$P = \begin{cases} \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j \frac{i!j!}{(i+j+1)!} t^{i+j+1} & 0 < t < T_1 \\ \dots & \\ \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j \frac{i!j!}{(i+j+1)!} t^{i+j+1} + & \\ \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j \frac{i!j!}{(i+j+1)!} (t-T_{z-1})^{i+j+1} - & \\ \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j \frac{i!j!}{(i+j+1)!} (T_z - t)^{i+j+1} & T_z < t < 2T_z \\ \dots & \\ 0 & t > T_m \end{cases} \tag{8}$$

Proof. Assume that Eq.6 holds, and P is defined as

$$P = p_t * p_t \tag{9}$$

then Eq.7 can be re-written as

$$\begin{aligned} \mathcal{L}\{P\} &= \mathcal{L}\{\sum_{i=1}^{n} a_i t^i\} \cdot \mathcal{L}\{\sum_{i=1}^{n} a_i t^i\} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j \frac{i!j!}{(i+j+1)!} \frac{(i+j+1)!}{s^{i+j+1+1}} \\ P &= \mathcal{L}^{-1}\{\sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j \frac{i!j!}{(i+j+1)!} \frac{(i+j+1)!}{s^{i+j+1+1}}\} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j \frac{i!j!}{(i+j+1)!} t^{i+j+1} \end{aligned} \tag{10}$$

The result of Eq.10 represents the polynomial library function, if no intersection in between during the convolution occurs. In case of intersections during the convolution, the area changes in between need to be considered. Therefore, the number of intersections z is related to the number of existing global extrema of the polynomial itself. In order to obtain the result for the convolution in between of intersections, Eq.10 can be re-written as

$$\begin{aligned} P &= \int_{t-T_{z-1}}^{T_z} p(u)p(t-u)du \\ &= \int_0^t p(u)p(t-u)du + \int_{t-T_{z-1}}^{0} p(u)p(t-u)du - \int_t^{T_z} p(u)p(t-u)du \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j \frac{i!j!}{(i+j+1)!} t^{i+j+1} + \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j \frac{i!j!}{(i+j+1)!} (t - T_{z-1})^{i+j+1} - \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j \frac{i!j!}{(i+j+1)!} (T_z - t)^{i+j+1} \end{aligned} \tag{11}$$

where T_z denotes the time stamp of the intersection. Concluding the results from Eq.10 and Eq.11 and replacing the result from Eq.11 for arbitrary intersections of the convolution product denoted with ... leads to the final solution

$$P = \begin{cases} \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j \frac{i!j!}{(i+j+1)!} t^{i+j+1} & 0 < t < T_1 \\ \dots & \\ \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j \frac{i!j!}{(i+j+1)!} t^{i+j+1} + & \\ \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j \frac{i!j!}{(i+j+1)!} (t-T_{z-1})^{i+j+1} - & \\ \sum_{i=0}^{n}\sum_{j=0}^{n} a_i a_j \frac{i!j!}{(i+j+1)!} (T_z - t)^{i+j+1} & T_z < t < 2T_z \\ \dots & \\ 0 & t > T_m \end{cases} \tag{12}$$

Q.E.D

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For demonstration purpose, Figure 2 shows the convolution behaviour for a polynomial of third order.

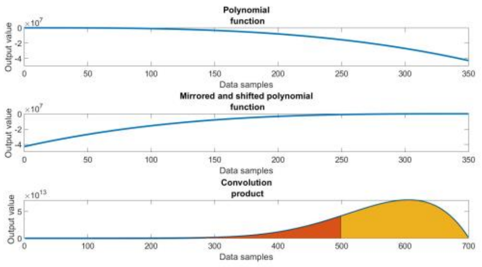

Figure 2: Convolution product of two polynomials of third order

2.2.2.Convolution operation for exponential identification

Having derived the relationship of the convolution operation between two polynomials, it is in interest to investigate the convolution behaviour of exponentials and its result, since the behaviour of both functionalities in time domain can be identical. Therefore, the general exponential expression s_t given as

$$s_t = a_1 \exp(bt) \tag{13}$$

with the given correspondence from Eq.7 will be investigated.

Theorem 2. Using Eq.13, the convolution of two polynomials itself results into a piecewise higher order exponential S with two intersections, which is

$$S = \begin{cases} a_1{}^2 t \exp(bt) & 0 < t < T \\ a_1{}^2\left[T\exp(bT) - (t-T)\exp\big(b(t-T)\big)\right] & \mathrm{T} < \mathrm{t} < 2\mathrm{T} \\ 0 & t > 2T \end{cases} \tag{14}$$

Proof. This theorem can be proved following a similar approach from theorem 1. Using the inverse Laplace transform from the convoluted product, it can be obtained that

$$\begin{aligned} L\{S\} &= L\{a_1 \exp(bt)\} \cdot L\{a_1 \exp(bt)\} \\ S &= a_1{}^2 t \exp(bt) \end{aligned} \tag{15}$$

To determine the convolution product for one intersecting area, the expression from Eq.11 is used with the result from Eq.15 and modified ranges, that is,

$$\begin{aligned} S &= \int_{t-T}^{T} s(u)s(t-u)du \\ &= a_1{}^2\left[T\exp(bT) - (t-T)\exp\big(b(t-T)\big)\right] \end{aligned} \tag{16}$$

Concluding the results from Eq.15 and Eq.16, the overall result for the convolution operation can be summarized as

$$S = \begin{cases} a_1{}^2 t \exp(bt) & 0 < t < T \\ a_1{}^2\left[T\exp(bT) - (t-T)\exp\big(b(t-T)\big)\right] & \mathrm{T} < \mathrm{t} < 2\mathrm{T} \\ 0 & t > 2T \end{cases} \tag{17}$$

Q.E.D

From Eq.12 and 17, it can be shown for the identification of the functional structure, that both the exponential and polynomial features can be distinguished, and two general structure expressions are provided. Thus, the convolution part of Eq.2 can be updated with the obtained results.

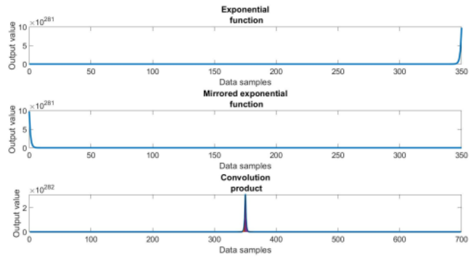

Figure 3: Convolution product of two exponential functions

2.3. Asymptotic analysis

Unless, the residual evaluation is revealed to be a non-white sequence after the application of the proposed two methods, this is referred to a misdetection of an unknown nonlinear candidate due to the numerical constraints of the Laplace operation. Therefore, the asymptotic analysis is proposed to focus on the sigmoid functions and rational functions without definition gaps.

Theorem 3. The properties of any sigmoid and rational function can be determined and distinguished by obtaining the related asymptote and the first two derivatives.

Proof. Let the generalised sigmoid function be

$$f(t) = \frac{L}{1+\exp(-k(t-t_0))} \tag{18}$$

and the rational function be

$$f(t) = \frac{K}{\sqrt[n]{T+t^p}} \text{ with } K, L \in \Re, n \in \Re^+ \backslash \{0\}, p \in \Re_{\mathrm{odd}} \tag{19}$$

Computing the first derivative and the related asymptotes for Eq.18 and 19 lead to

$$f(t) = \frac{L}{1+\exp(-k(t-t_0))} = \frac{L\exp\big(k(t-t_0)\big)}{1+\exp\big(k(t-t_0)\big)} \tag{20}$$

$$\lim_{t\to+\infty} f(t) = L \text{ and } \lim_{t\to-\infty} f(t) = 0 \tag{21}$$

$$\frac{df}{dt} = k\, f(t)(1-f(t)) \tag{22}$$

$$f(t) = \frac{K}{\sqrt[n]{T+t^p}} \tag{23}$$

$$\lim_{t\to+\infty} f(t) = 0 \text{ and } \lim_{t\to-\infty} f(t) = 0 \tag{24}$$

$$\frac{df}{dt} = \frac{-K}{n\sqrt[n]{(T+t^p)}^{n+1}} p t^{p-1} \qquad \text{Q.E.D} \tag{25}$$

As a result, it can be obtained that the characteristics of a sigmoid function and a rational function can be defined and distinguished between the asymptotes and its first derivative. Finally, the derived theoretical results can be added to the regression indicator in Eq.2 to complete the structure identification.

3. SIMULATION

The derived structure identification method will be tested on the use case of an inverted pendulum, which is defined as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{g}{l}\sin(x) - \frac{k}{m}y \end{aligned} \tag{26}$$

where g is the gravitational constant, l the length of the rope, k the damping factor, and m the mass of the pendulum, while x and y are respectively the elongation angle and its deviaton with the following normed parameters:

Table 1: Parameter setting for the inverted pendulum process

Variable	g	l	k	m
Values	9.81	3	3	20

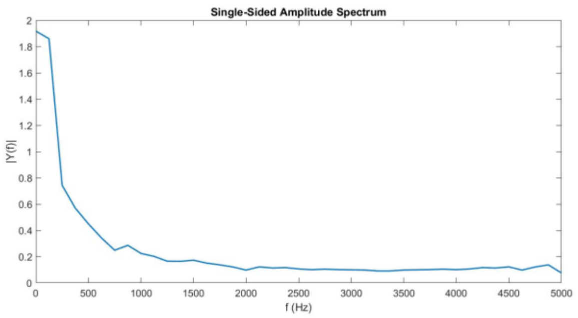

Figure 4: Fast Fourier Transform of output signal

Firstly, the system is excited with an PRBS signal to determine the dynamic content in the data set to apply the structure identification procedure. After ensuring a sufficient excitation, the Fast Fourier Transform is applied. From Figure 4, the specific frequencies at 850 Hz and 1500 Hz are obtained, which indicate a sinusoidal influence. Moreover, the residual shows a non-white sequence, which hints another dynamic influence in the dataset. Applying the convolution product provides a fit with the convolution product structure from Eq.12, that is,

$$S = \begin{cases} 0.002(x + 0.9586)^5 & 0 < x < 60 \\ 0.002[60-(x + 0.9586)^5] & 60 < x < 107 \\ 0 & x > 107 \end{cases} \quad (27)$$

with the resulting polynomial

$$s_t = 0.2488x^2 + 0.2385x \quad (28)$$

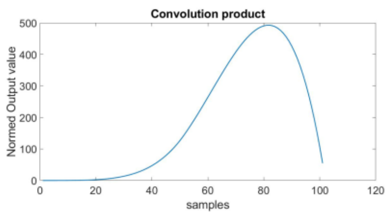

Figure 5: Convolution product of rest output signal

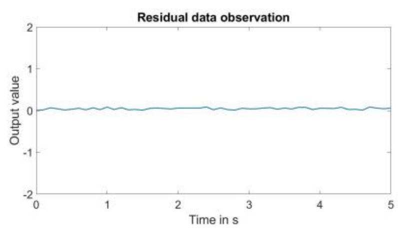

Figure 6: Residual evaluation after convolution operation

Since the last residual evaluation indicates a white-noise sequence with scores below the significance level according to the Ljung-Box Test statistics, the structure identification is completed. From the results, it can be obtained that the following nonlinear candidates need to be considered to achieve a proper parameter estimation:

Table 2: Selected nonlinear candidates

Functional candidates	$\sin(x,y)$	$\cos(x,y)$	x, y	$(x,y)^2$

Hence, the sparse regression is applied, and the following parameter estimates are determined:

Table 3: Parameter estimates

	$\sin(x)$	$\cos(x)$	y	x^2
$\dot{x}$	0	0	1	0
$\dot{y}$	3.27	0	-0.15	0

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= 3.27\sin(x) - 0.15y \end{aligned} \quad (29)$$

4. CONCLUSIONS

This paper proposed a new data-driven structure identification method to support and improve the nonlinear system identification procedure. The investigated structure identification method provided the appropriate candidates for the sparse regression method to determine the estimates for the nonlinear candidates for the SISO use case of an inverted pendulum. The sequential algorithm proved that it is capable of determining harmonic functions over the Fast Fourier Transform and covering the nonlinear dynamics over the convolution operation and asymptotic analysis as well with respect to the bijective feature of the candidates and a sufficient excitation with a PRBS input signal. Future work will focus on extending the developed structure identification method to noisy data and coupled nonlinearities.

REFERENCES

Shardt, Y. A. W. *Statistics for Chemical and Process Engineers*. Springer International Publishing AG, Switzerland, 2015

Unbehauen, H. & Bohn, C. *Identification of dynamical systems: methods for experimental modelling from measurements*. Springer Fachmedien GmbH, Wiesbaden 2016.

Narendra, K. S. & Parthasarathy, K. *Identification and Control of Dynamical Systems Using Neural Networks*. IEEE Transactions of Neural Networks, Vol.1, No.1, March 1990.

Brunton, S. L., Proctor, J. L., Kutz, N. *Discovering governing equations from data: Sparse identification of nonlinear dynamical systems*. arXiv:1509.03580v1, 11. September 2015.

Nelles, O. *Nonlinear System Identification: From classical approaches to Neural Networks and Fuzzy Models*. Springer Verlag Berlin Heidelberg New York, 2001.

Hofmann, S. *Identification of nonlinear mechtronic systems based on Volterra series*. Prentice Hall, 2nd edition, 1999.