Preprints, 6th IFAC Conference on Nonlinear Model Predictive Control
Madison, WI, USA, August 19-22, 2018

Contract-based Predictive Control for Modularity in Hierarchical Systems

Tobias Bäthge * **Markus Kögel** * **Stefano Di Cairano** ** **Rolf Findeisen** *

* *Laboratory for Systems Theory and Automatic Control, Otto von Guericke University, Magdeburg, Germany.*
{tobias.baethge, markus.koegel, rolf.findeisen}@ovgu.de

** *Mitsubishi Electric Research Laboratories (MERL), Cambridge, MA, USA.*
dicairano@ieee.org

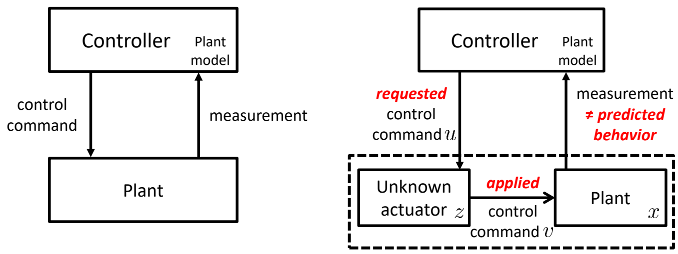
Abstract: Hierarchical control architectures pose challenges for control, as lower-level dynamics, such as from actuators, are often unknown or uncertain. If not considered correctly in the upper layers, the requested and the applied control signals will differ. Thus, the actual and the predicted plant behavior will not match, likely resulting in constraint violation and decreased control performance. We propose a model predictive control scheme in which the upper and lower levels—the controller and the actuator—agree on a "contract" that allows to bound the error due to neglected dynamics. The contract allows to guarantee a desired accuracy, enables modularity, and breaks complexity: Components can be exchanged, vendors do not need to provide in-depth insights into the components' working principle, and complexity is reduced, as upper-level controllers do not need full model information of the lower level—the actuators. The approach allows to consider uncertain actuator dynamics with flexible, varying sampling times. We prove repeated feasibility and input-to-state stability and illustrate the scheme in an example for a hierarchical controller/plant cascade.

Keywords: Hierarchical control; predictive control; modularization; contracts; robustness.

1. INTRODUCTION

Control problems in modern applications like Automatic Driving or the "Internet of Things" are often highly interconnected, as they involve combinations of supervisory controllers and lower-level actuators, see e.g. Campbell et al. (2010); Di Cairano and Borrelli (2016); Lucia et al. (2016). Such hierarchical structures, spanning multiple levels, often consist of controllers, sensors, and actuators from different manufacturers. Combining components from multiple vendors often has an impact on the available knowledge of and the communication between these components: Exact dynamics of subsystems or neighboring components might be unknown, as companies want to protect proprietary knowledge, or might change when replacing a component with a model from a different vendor. Designing model-based controllers without such knowledge is difficult, especially in the case of Model Predictive Control (MPC). MPC schemes rely on sufficiently correct and detailed models for the prediction of the future system behavior, see e.g. Maciejowski (2002); Rawlings et al. (2017); Mayne (2014)), to achieve good control performance.

We consider the case of unknown actuator dynamics, as shown in Fig. 1, to outline the appearing challenges and



(a) Simple hierarchy with exact plant knowledge.

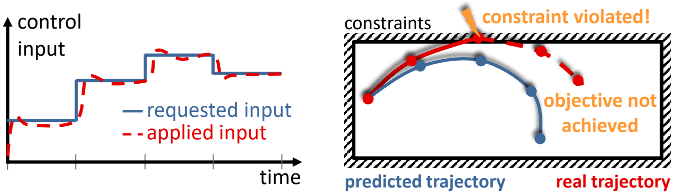
(b) Unknown actuator dynamics introduce model-plant mismatch.

Fig. 1. Nominal plant structure and hierarchical structure including an unknown actuator.

provide an MPC strategy that overcomes these, enables modularity, allows privacy between the components, and breaks complexity. The unknown or uncertain dynamics can "slow down", delay, or modify the requested control input, leading to a real input that is different to the one that the controller commanded. As a consequence, the desired optimal behavior will not be reached and constraints might be violated, cf. Fig. 2. Neglecting this mismatch can result in increased conservatism, higher energy consumption, or even instability.

We propose an MPC scheme that takes the additional dynamics—without in-depth knowledge—in form of an additional uncertainty into account. The uncertainty is bounded by the maximum error or an accuracy around

Major parts of this work where developed during a research stay of TB and RF at MERL. TB, MK, and RF are members of the International Max Planck Research School for Advanced Methods in Process and Systems Engineering (IMPRS), Magdeburg, Germany. TB and MK contributed equally to this work.



(a) Requested (solid, blue line) and applied input, due to the additional actuator dynamics (dashed, red line).

(b) Degraded controller performance: Accuracy is lost, constraints can be violated, and control energy might be increased.

Fig. 2. Influence of unknown actuator dynamics on controller performance and constraint satisfaction.

a "nominal" solution, requested by the controller, and for the actuator to guarantee. Achieving the desired accuracy error despite the actuator dynamics demands either a reduced range of feasible control inputs or control input changes, or an increased or decreased control input change frequency. The actuator, as an "active" component, provides this information to the controller in form of a "contract", by calculating an upper bound of the resulting mismatch between the nominal and real dynamics.

The agreement of an accuracy and corresponding input constraints forms a contract between the controller and the actuator. This allows to construct a modular control structure: The internal behavior of a control loop component can change, e.g. when an actuator is exchanged for a different type, model, or brand, as long as the agreed-on contract is enforced. Furthermore, the overall complexity is reduced, as the upper level does not need a detailed model of the lower-level actuator. It also enables hiding proprietary information, which vendors often demand.

Contract-based MPC schemes allow to tackle the problem for complex systems, cf. Lucia et al. (2015) and the references therein. We focus here on the hierarchical case, i.e., we consider contracts for accuracy and input constraints between components on different hierarchy levels. In comparison to the general case, our contracts are static, i.e., they do not involve a variation in time. We employ a discrete-time formulation, with the flexibility of different sampling times of controller and actuator, and use a robust, tube-based MPC scheme that allows an expansion towards additional model uncertainty. Summarizing, the contribution of this work is an MPC scheme that

- allows considering additional unknown and uncertain actuator dynamics, by agreeing on a contract, consisting of accuracy bounds and input (rate) constraints,
- guarantees constraint satisfaction, employing a tube-based robust MPC controller, and
- is formulated in a discrete-time setting allowing for different sampling times for actuator and controller.

Such an approach of "modularity in hierarchy", where model complexity is covered by an uncertainty description, can be compared and further expanded to exploiting the granularity (see e.g. Bäthge et al. (2016)) of different system models over the prediction, where different model complexities are captured by an extra uncertainty. Similar ideas also exist for exploiting model reduction approaches and decentralized MPC without communication, e.g. Kögel and Findeisen (2015, 2018).

The remainder of this paper is structured as follows: The general framework is introduced in Section 2. Section 3 presents an approach to bound the error that is introduced by the additional actuator dynamics in the control loop. The design of a robust MPC scheme that exploits the bound in form of a contract is presented in Section 4. A simulation example is presented in Section 5. Section 6 closes with a summary and suggestions for future work.

We use standard notation: For two sets $\mathbb{A}$, $\mathbb{B}$ and a matrix M, $\mathbb{A} \oplus \mathbb{B}$, $\mathbb{A} \ominus \mathbb{B}$, $M\mathbb{A}$, $\times$ denote the Minkowski sum, the Minkowski difference, the set multiplication, and the Cartesian product, respectively, see e.g. Blanchini and Miani (2015). A set $\mathbb{S}$ is called robust positive invariant (RPI) under $s_{k+1} = Ss_k + e_k$, $e_k \in \mathbb{E}$, where $\mathbb{E}$ is a convex, compact set with $0 \in \mathbb{E}$, if $\forall s_k \in \mathbb{S}$, $e_k \in \mathbb{E}$: $s_{k+1} \in \mathbb{S}$. $a_{i|j}$ denotes the value of a at time t_i, calculated at time t_j.

2. HIERARCHICAL CONTROL ARCHITECTURE

We consider a hierarchical control architecture in which an upper-level controller interacts with a plant and a lower-level actuator. The plant dynamics, which the upper-level controller is based on, cf. Fig. 1, are given by

$$\dot{x}(t) = Ax(t) + Bv(t), \qquad y(t) = Cx(t) + Dv(t), \qquad (1)$$

where $x(t) \in \mathbb{R}^{n_x}$ denotes the states, $v(t) \in \mathbb{R}^{n_u}$ the inputs applied by the actuator, and $y(t) \in \mathbb{R}^{n_y}$ the plant outputs. The plant states, inputs, and outputs need to fulfill state, input, and output constraints,

$$x(t) \in \mathbb{X}, \qquad v(t) \in \mathbb{U}, \qquad y(t) \in \mathbb{Y}, \qquad (2)$$

respectively. We assume that $\mathbb{X}$, $\mathbb{U}$, and $\mathbb{Y}$ are compact, convex polytopes containing the origin in their interior.

The controller requests a command signal $u(t) \in \mathbb{U}$ from the actuator, which results in the applied input $v(t)$. Ideally,

$$v(t) = u(t), \qquad (3)$$

i.e., the requested and the applied input signals match. However, in reality, the actuator often introduces additional dynamics, which we consider to be of the form

$$\dot{z}(t) = A_a z(t) + B_a u(t), \qquad v(t) = C_a z(t). \qquad (4)$$

Here, $z(t) \in \mathbb{R}^{n_z}$ denotes the actuator states, $u(t) \in \mathbb{R}^{n_u}$ the inputs commanded to the actuator, and $v(t) \in \mathbb{R}^{n_u}$ the actuator outputs that are applied to the plant as inputs. We assume that the actuator satisfies:

Assumption 1 (Stable actuator dynamics): The actuator dynamics (4) are asymptotically stable, i.e., A_a is Hurwitz.

Assumption 2 (Unity gain of the actuator dynamics): The actuator has a steady state unity gain: A constant $u(t)$ implies $v(t) \to u(t)$ and $\forall u^s : 0 = A_a z^s + B_a u^s$, $u^s = C_a z^s$.

Note that Assumption 2 can be satisfied easily by a suitable choice/scaling of the inputs $u(t)$.

Remark 3 (Actuator constraints): For simplicity and as we assume a well-behaving and stable actuator, we only consider input constraints $\mathbb{U}$ for the actuator, but no constraints on the actuator state.

As outlined, the full lower-level actuator dynamics might be unknown or unavailable, e.g. for proprietary or privacy reasons. Thus, the upper-level controller has no direct knowledge of the actuator model (4). We propose an approach where it can negotiate a "contract" with the actuator during the design or operation phase, which specifies input changes and input limits to achieve a desired actuator-plant error.

Overall, we want to solve the following problem:

Problem 4 (Modular contract-based controller design): Design an upper-level controller that robustly stabilizes the lower-level plant (1) *and achieves constraint satisfaction despite limited knowledge of the actuator dynamics.*

To tackle this problem, we suggest that the controller treats the unknown actuator dynamics as an additional uncertainty that is considered in the prediction. The bounding set for the uncertainty, the accuracy, together with the necessary constraints on the control input form an actuator-controller contract. To calculate these sets, we derive bounds for the error that is introduced by the additional dynamics.

Note that, for clarity of presentation, we focus on a single actuator-controller configuration. Expansions to multiple-actuator cases are easily possible.

3. BOUNDING THE ACTUATOR ERROR

The contract between controller and actuator is based on a bounding set for the error that the actuator dynamics (4) cause in comparison to the ideal actuator dynamics (3). We assume that the upper-level controller is a sampled-data controller based on a discrete-time model, while the plant itself operates in continuous time. To this end, we introduce discrete-time formulations for the combined systems used in the controller, which allow to bound the resulting error. Applying standard discretization techniques, cf. Ogata (1995), using a sampling time $T > 0$ and a zero-order hold $u(t) = u(t_k)$, for $t \in [kT, kT+T)$, $k = 0, 1, \ldots$, leads to the following discrete-time models:

Model of the plant with an ideal actuator: The ideal system, consisting of the plant (1) and the ideal actuator (3), is given by

$$\hat{x}(t_{k+1}) = \hat{F}\hat{x}(t_k) + \hat{G}u(t_k) \tag{5a}$$
$$\hat{y}(t_k) = C\hat{x}(t_k) + Du(t_k). \tag{5b}$$

Here, $\hat{\cdot}$ denotes the variables of the combination of plant and ideal actuator, with $\hat{F} = \mathrm{e}^{AT}$. This model, expanded by an error term, is used in the upper-level controller.

Model of the plant and the actuator: The real system, combining the plant (1) and the actuator (4), is given by

$$\dot{\xi}(t) = \begin{pmatrix} A & BC_\mathrm{a} \\ 0 & A_\mathrm{a} \end{pmatrix} \xi(t) + \begin{pmatrix} 0 \\ B_\mathrm{a} \end{pmatrix} u(t), \quad \xi = \begin{pmatrix} x \\ z \end{pmatrix}. \tag{6}$$

The corresponding discrete-time system, assuming a sampling time T, is given by

$$\xi(t_{k+1}) = \begin{pmatrix} F & F_\mathrm{c} \\ 0 & F_\mathrm{a} \end{pmatrix} \xi(t_k) + \begin{pmatrix} G_x \\ G_z \end{pmatrix} u(t_k) \tag{7a}$$
$$y(t_k) = (C \;\; 0)\, \xi(t_k) + Du(t_k), \tag{7b}$$

where $F = \mathrm{e}^{AT}$ and $F_\mathrm{a} = \mathrm{e}^{A_\mathrm{a}T}$. We furthermore know:

Proposition 5 (Basic properties of the discretized systems): If Assumption 1 holds, then F_a is Schur stable. There exists a bounded RPI set $\mathbb{Z}$ for the actuator dynamics, such that

$$\mathbb{Z} \subseteq F_\mathrm{a}\mathbb{Z} \oplus G_z\mathbb{U}, \tag{8}$$

and the steady state z_∞ of the actuator for a given steady state input u_∞ is given by

$$z_\infty = (I - F_\mathrm{a})^{-1} G_z u_\infty. \tag{9}$$

If Assumptions 1 and 2 hold, then

$$\Delta G = G_x - \hat{G} = -F_\mathrm{c}(I - F_\mathrm{a})^{-1} G_z, \tag{10}$$

and for any constant $u(t) = u_\mathrm{c}$ $v(t) = u(t) = u_\mathrm{c}$ and $Bv(t) = Bu(t) = Bu_\mathrm{c}$, and $\hat{G}u_\mathrm{c} = G_x u_\mathrm{c} + F_\mathrm{c}(I - F_\mathrm{a})^{-1} G_z u_\mathrm{c}$.

The results are derived for a fixed sampling time T.

Remark 6 (Flexible sampling time): In principle, there is no need to use a constant input signal for the full sampling time T in the actuator. One could use a faster input sampling, e.g. $h = \frac{T}{H}$, $H \in \mathbb{N}$, allowing a faster actuator controller of the form

$$u^\mathrm{c}(t) = \begin{cases} K_1 \begin{pmatrix} z(t_k) \\ u(t_k) \end{pmatrix}, & t \in [t_k, t_k + h) \\ K_2 \begin{pmatrix} z(t_k + h) \\ u(t_k + h) \end{pmatrix}, & t \in [t_k + h, t_k + 2h) \\ \vdots & . \end{cases} \tag{11}$$

Here, u^c is the command sent to the plant, which can be reformulated similarly by (7).

The controller does not know the actuator dynamics (4). Thus, the predicted state at the next time instant, $\hat{x}(t_{k+1})$, in general, does not match the real state $x(t_{k+1})$, even if $\hat{x}(t_k) = x(t_k)$. To account for this mismatch, we introduce an artificial disturbance $w(t_k)$ to capture the resulting error. We aim to find a set $\mathbb{W}$ which allows to bound the behavior of the system, such that

$$\begin{aligned} \exists w(t_k) \in \mathbb{W} : x(t_{k+1}) &= Fx(t_k) + F_\mathrm{c} z(t_k) + G_x u(t_k) \\ &= \underbrace{\hat{F}x(t_k) + \hat{G}u(t_k)}_{=\hat{x}(t_{k+1})} + w(t_k). \end{aligned} \tag{12}$$

In other words, the set-based dynamics

$$\tilde{x}(t_{k+1}) = \hat{F}\tilde{x}(t_k) + \hat{G}u(t_k) + w(t_k), \quad w(t_k) \in \mathbb{W}, \tag{13}$$

outer-bound the behavior of the real system.

We establish a way to find such a set $\mathbb{W}$ and that the error approaches zero as the input goes to zero:

Proposition 7 (Bounding the actuator error): Under Assumption 1 and if $z(t_0) \in \mathbb{Z}$, where $\mathbb{Z}$ satisfies (8), *a $\mathbb{W}$ such that* (12) *holds is given by*

$$\mathbb{W} = F_\mathrm{c}\mathbb{Z} \oplus \Delta G \mathbb{U}. \tag{14}$$

Furthermore, if $u(t_k) \to 0$, then $w(t_k) \to 0$.

Proof. Exploiting (12) and Proposition 5, it is clear that

$$w(t_k) = x(t_{k+1}) - \hat{x}(t_{k+1}) = F_\mathrm{c} z(t_k) + \Delta G u(t_k).$$

This yields equation (14) using the bounds on $z(t_k)$ and $u(t_k)$. Note that F_a is Schur stable. Tens, $u(t_k) \to 0$ in (7) implies $z(t_k) \to 0$, thus, $F_\mathrm{c} z(t_k) + \Delta G u(t_k) \to 0$. □

Therefore, the error w caused by the actuator vanishes for small inputs u. Note that the error caused by the actuator can be large.

Restricting the rate of input changes,

$$\Delta u(t_k) = u(t_k) - u(t_{k-1}) \in \Delta\mathbb{U}, \tag{15}$$

with a compact, convex polytope $\Delta\mathbb{U}$ containing the origin, allows to further decrease the actuator uncertainties.

For input change constraints $\Delta\mathbb{U}$ one can derive similar results to Proposition 7. Defining the difference between actuator state steady state at t_i and the applied input $u(t_{i-1})$,

$$\Delta z(t_i) = z(t_i) - (I - F_{\mathrm{a}})^{-1} G_z u(t_{i-1}), \tag{16}$$

allows to establish the following:

Proposition 8 (Error in Δu formulation): Let Assumptions 1 and 2 and $\Delta z(t_0) \in \Delta\mathbb{Z}$ hold, where $\Delta\mathbb{Z}$ satisfies

$$\Delta\mathbb{Z} \subseteq F_{\mathrm{a}}\Delta\mathbb{Z} \oplus -F_{\mathrm{a}}(I - F_{\mathrm{a}})^{-1} G_z \Delta\mathbb{U}. \tag{17}$$

If $\Delta u(t_i) \in \Delta\mathbb{U}$, $i = 0, \dots, k$, then a $\mathbb{W}$ such that (12) is satisfied is given by

$$\mathbb{W} = F_{\mathrm{c}}\Delta\mathbb{Z} \oplus \Delta G \Delta\mathbb{U}. \tag{18}$$

Proof. Proposition 5 and (16) yield

$$\begin{aligned} \Delta G u(t_{i-1}) &= -F_{\mathrm{c}}(I - F_{\mathrm{a}})^{-1} G_z u(t_{i-1}) \\ &= F_{\mathrm{c}}(\Delta z(t_i) - z(t_i)). \end{aligned} \tag{19}$$

Thus, we obtain from (12) that

$$\begin{aligned} w(t_i) &= F_{\mathrm{c}} z(t_i) + \Delta G(u(t_{i-1}) + \Delta u(t_i)) \\ &= F_{\mathrm{c}}\Delta z(t_i) + \Delta G \Delta u(t_i), \end{aligned} \tag{20}$$

establishing (18) using the bounds on $\Delta z(t_i)$ and $\Delta u(t_i)$.

Using (7), (22), and (16), we obtain

$$\begin{aligned} \Delta z(t_{i+1}) &= F_{\mathrm{a}} z(t_i) + G_z u(t_i) - (I - F_{\mathrm{a}})^{-1} G_z u(t_i) \\ &= F_{\mathrm{a}} z(t_i) - F_{\mathrm{a}}(I - F_{\mathrm{a}})^{-1} G_z u(t_i) \\ &= F_{\mathrm{a}}\Delta z(t_i) - F_{\mathrm{a}}(I - F_{\mathrm{a}})^{-1} G_z \Delta u(t_i), \end{aligned} \tag{21}$$

leading to (17). □

Thus, to achieve a desired actuator accuracy, input and input rate constraints need to be satisfied by the upper-level controller. To this end, we rewrite the input $u(t_k)$ as a sum of input *changes*:

$$u(t_k) = u(t_{-1}) + \sum_{i=0}^{k} \Delta u(t_i). \tag{22}$$

Here, $u(t_{-1}) = \lim_{t\to 0^-} z(t)$ is the actuator state just before $t = 0$.

Given this, a bounding set $\mathbb{W}$ can be obtained from (14). $\mathbb{W}$ then needs to be equal or inside the desired accuracy bound $\mathbb{W}_{\mathrm{request}}$ that the upper-level controller requests. During the contract negotiation, as outlined in Fig. 3, the actuator obtains a request $\mathbb{W}_{\mathrm{request}}$ from the controller. This can be rejected or confirmed by suitable sets $\mathbb{U}$, $\Delta\mathbb{U}$, and $\mathbb{W}$. The controller can then certify whether it accepts the sets $\mathbb{U}$ and $\Delta\mathbb{U}$ or it can ask for a different $\mathbb{W}_{\mathrm{request}}$.

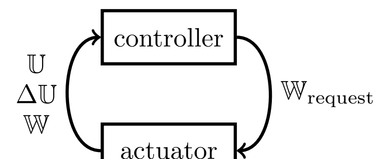


Fig. 3. Contract negotiation.

Remark 9 (Uncertain actuator dynamics): We assume that no uncertainties are present in the actuator. A straightforward extension to cover "multiplicative uncertainties" is to assume that the matrices F_{a}, F_{c}, G_x, and G_z are given by

$$\sum_{i=1}^{V} \lambda_i F_{\mathrm{a},i}, \quad \sum_{i=1}^{V} \lambda_i F_{\mathrm{c},i}, \quad \sum_{i=1}^{V} \lambda_i G_{x,i}, \quad \sum_{i=1}^{V} \lambda_i G_{z,i}, \tag{23}$$

respectively, where $\lambda_i \geq 0$ and $\sum_{i=1}^{V} \lambda_i = 1$. $F_{\mathrm{a},i}$, $F_{\mathrm{c},i}$, $G_{x,i}$, and $G_{z,i}$ form the corners defining the uncertainty.

In this case, $\mathbb{W} \subseteq \mathbb{W}_i$, $i = 1, \dots, V$, where $\mathbb{W}_i$ are the sets obtained from Proposition 7 (or Proposition 8) for the matrices $F_{\mathrm{a},i}$, $F_{\mathrm{c},i}$, $G_{x,i}$, and $G_{z,i}$. This allows to determine a bound on $w(t_k)$ such that equation (12) holds for (23).

4. CONTROLLER DESIGN

For the upper-level controller, a robust, tube-based approach, cf. Mayne et al. (2005, 2006).

To guarantee input rate constraints satisfaction, the dynamics (5) are augmented,

$$\tilde{x} = \begin{pmatrix} \hat{x}_k \\ u_{k-1} \end{pmatrix}, \quad \tilde{y} = \begin{pmatrix} y_k \\ u_k - u_{k-1} \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & -I \end{pmatrix} \tilde{x} + \begin{pmatrix} D \\ I \end{pmatrix} u,$$

and the constraints $\tilde{\mathbb{Y}} = \mathbb{Y} \times \Delta\mathbb{U}$ are enforced. Here, $\mathbb{Y}$ are general "output" constraints.

The resulting tube-based MPC scheme repeatedly solves

$$\min_{\mathbf{x}_k, \mathbf{u}_k} J(\mathbf{x}_k, \mathbf{u}_k), \tag{24}$$

where

$$\mathbf{x}_k = \left\{x_{k|k}, \dots, x_{k+N|k}\right\}, \tag{25a}$$

$$\mathbf{u}_k = \left\{u_{k|k}, \dots, u_{k+N-1|k}\right\}, \tag{25b}$$

and $N \in \mathbb{N}$, $N > 1$, is the length of the prediction horizon, and where the cost function is given by

$$J(\mathbf{x}_k, \mathbf{u}_k) = x_{k+N|k}^{\mathsf{T}} P x_{k+N|k} + \sum_{i=k}^{k+N-1} x_i^{\mathsf{T}} Q x_i + u_i^{\mathsf{T}} R u_i \tag{25c}$$

with positive definite weighting matrices Q, P, and R. The dynamics and constraints are given by

$$x_{i+1|k} = \hat{F} x_{i|k} + \hat{G} u_{i|k},\ i = k, \dots, k+N-1, \tag{25d}$$

$$x_{k|k} = \{x(t_k)\} \oplus \mathbb{D}, \tag{25e}$$

$$x_{i|k} \in \tilde{\mathbb{X}},\ i = k, \dots, k+N-1, \tag{25f}$$

$$u_{i|k} \in \tilde{\mathbb{U}},\ i = k, \dots, k+N-1, \tag{25g}$$

$$C x_{i|k} + D u_{i|k} \in \tilde{\mathbb{Y}},\ i = k, \dots, k+N-1, \tag{25h}$$

$$x_{k+N|k} \in \mathbb{T}. \tag{25i}$$

$\mathbb{D}$ describes the constraint back-off that provides a safety margin, $\tilde{\mathbb{X}}$, $\tilde{\mathbb{U}}$, and $\tilde{\mathbb{Y}}$ are tightened constraint sets for states, inputs, and outputs, and $\mathbb{T}$ is the terminal set.

The input that is commanded to the actuator is given by

$$u(t_k) = u_{k|k}^{\star} + K(x(t_k) - x_{k|k}^{\star}). \tag{26}$$

Here, $\cdot^{\star}$ denotes the optimal solution of the optimization problem (25) and K is the local feedback used in the tube controller.

For robust stability and constraint satisfaction, one then needs:

Assumption 10 (Conditions on sets $\mathbb{D}$, $\tilde{\mathbb{X}}$, $\tilde{\mathbb{U}}$, $\tilde{\mathbb{Y}}$, and $\mathbb{T}$): The constraint back-off set $\mathbb{D}$ satisfies

$$\mathbb{D} \supseteq (\hat{F} + \hat{G}K)\mathbb{D} \oplus \mathbb{W}. \tag{27}$$

The sets $\tilde{\mathbb{X}}$, $\tilde{\mathbb{U}}$, and $\tilde{\mathbb{Y}}$ contain a neighborhood of the origin and satisfy

$$\tilde{\mathbb{X}} = \mathbb{X} \ominus \mathbb{D}, \quad \tilde{\mathbb{U}} = \mathbb{U} \ominus K\mathbb{D}, \quad \tilde{\mathbb{Y}} = \mathbb{Y} \ominus (C + DK)\mathbb{D}. \tag{28}$$

$\mathbb{T}$ contains a neighborhood of its origin and satisfies

$$(\hat{F} + \hat{G}K)\mathbb{T} \subseteq \mathbb{T}, \; \mathbb{T} \subseteq \tilde{\mathbb{X}}, \; C\mathbb{T} + D\mathbb{T} \subseteq \tilde{\mathbb{Y}}, \; K\mathbb{T} \subseteq \tilde{\mathbb{U}}. \tag{29}$$

For obtaining suitable sets, one can use several tools, see Herceg et al. (2013) and Riverso et al. (2013). Furthermore, we demand:

Assumption 11 (Conditions on the tube control gain): The control gain K is such that the control laws

$$u(t_k) = K\hat{x}(t_k), \qquad u(t_k) = Kx(t_k) \tag{30}$$

asymptotically stabilize systems (5) *and* (7), *respectively.*

Similar assumptions are exploited in works on decentralized MPC. The control gain K can be calculated using a linear-quadratic regulator (LQR) design.

Subject to the given assumptions, one can establish recursive feasibility:

Theorem 12 (Recursive feasibility): Let Assumptions 1, 10, and 11 and $z(t_0) \in \mathbb{Z}$ hold, where $\mathbb{Z}$ satisfies (8). *If input rate constraints are used, let $\Delta z(t_0) \in \Delta\mathbb{Z}$ hold, where $\Delta\mathbb{Z}$ satisfies* (17). *Then, the closed-loop system* (7), (26) *is recursively feasible, i.e., if* (25) *is feasible at t_0, then for any $k \geq 0$,* (25) *is feasible, $u(t_k) \in \mathbb{U}$, $x(t_k) \in \mathbb{X}$, and $y(t_k) \in \mathbb{Y}$.*

Proof. For clarity and due to space limitations, we only sketch the proof. First note that the conditions of Proposition 8 are satisfied. Thus, the closed-loop system satisfies (13). Following standard tube-based MPC arguments, we can guarantee recursive feasibility and constraint satisfaction robustly w.r.t. the artificial disturbance w. □

If the sets $\mathbb{D}$, $\tilde{\mathbb{X}}$, $\tilde{\mathbb{U}}$, $\tilde{\mathbb{Y}}$, and $\mathbb{T}$ are convex, closed polytopes, then (25) is a convex quadratic program, which can be solved efficiently, see e.g. Boyd and Vandenberghe (2004); Rawlings et al. (2017); Lucia et al. (2016). Alternatively, one might choose $\mathbb{D}$, $\tilde{\mathbb{X}}$, $\tilde{\mathbb{U}}$, $\tilde{\mathbb{Y}}$, and $\mathbb{T}$ as ellipsoids, in which case (25) is a convex quadratically-constained quadratic program, for which, by now, efficient, tailored solution approaches exist, as well.

To establish stability, we need further assumptions on the terminal penalty in the cost function:

Assumption 13 (Terminal penalty): The terminal penalty P in the cost function (25c) *satisfies*

$$P = (\hat{F} + \hat{G}K)^{\mathsf{T}} P (\hat{F} + \hat{G}K) + Q + K^{\mathsf{T}} R K. \tag{31}$$

Exploiting Theorem 12, one can then establish stability:

Theorem 14 (Asymptotic stability): Let Assumptions 1, 10, 11, and 13 and $z(t_0) \in \mathbb{Z}$ hold, where $\mathbb{Z}$ satisfies (8). *If* (25) *is feasible at t_0, then the closed-loop system* (7), (26) *is asymptotically stable.*

Proof. *(Sketch).* Theorem 12 guarantees recursive feasibility and constraint satisfaction. Moreover, with the assumptions made for $x_{0|0} = x(t_0)$, the LQR controller is locally admissible. Together with the fact that all constraint sets in the optimization problem (25) are compact, $\exists c_1 > 0$ s.t. $\|x^\star_{0|0}\| \leq c_1 \|x(t_0)\|$ and $\|u^\star_{0|0}\| \leq c_1 \|x(t_0)\|$. This allows, similar to standard tube-based MPC, to show that the nominal state $x^\star_{k|k}$ is asymptotically stable.

Considering the dynamics of the plant and actuator, we can furthermore derive that

$$\Delta\xi(t_k) = \xi(t_k) - \begin{pmatrix} x^\star_{k|k} \\ 0 \end{pmatrix}, \tag{32a}$$

$$\Delta\xi(t_{k+1}) = \tilde{F}\Delta\xi(t_k) + \tilde{G}u^\star_{k|k} + \begin{pmatrix} \Delta x^\star \\ 0 \end{pmatrix}, \tag{32b}$$

where

$$\Delta x^\star = \hat{F}x^\star_{k|k} + \hat{G}u^\star_{k|k} - x^\star_{k+1|k+1},$$

$$\tilde{F} = \begin{pmatrix} F & F_c \\ 0 & F_a \end{pmatrix} + \begin{pmatrix} G_x \\ G_z \end{pmatrix} (K \;\; 0), \qquad \tilde{G} = \begin{pmatrix} G_x - \hat{G} \\ G_z \end{pmatrix},$$

and where $\tilde{F}$ is Schur stable. Note that the $\Delta\xi$ dynamics are input-to-state stable (ISS) in terms of the nominal state and input, and $\exists c_2 > 0$ s.t. $\|\Delta\xi(t_0)\| \leq c_2 \left\| \left(x(t_0)^{\mathsf{T}} \; z(t_0)^{\mathsf{T}}\right)^{\mathsf{T}} \right\|$, which implies that $\Delta\xi$ is asymptotically stable. Thus, the overall system is stable. □

Summarizing, we have established repeated feasibility and stability of the complete controller-actuator-plant system.

5. SIMULATION EXAMPLE

We consider a double integrator, given by

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v(t), \quad y(t) = (1 \;\; 0)\, x(t). \tag{33}$$

The actuator is assumed to be of first order,

$$\dot{z}(t) = -20z(t) + 20u(t), \qquad v(t) = z(t). \tag{34}$$

Corresponding discrete-time formulations (5) and (7) are obtained using a sampling time $T = 0.3$, and states, control inputs, and control input changes are box-constrained:

$$\mathbb{X} = [-10, 10] \times [-10, 10], \; \mathbb{U} = [-2, 2], \; \Delta\mathbb{U} = [-0.4, 0.4].$$

The weighting matrices in the cost function are chosen as $Q = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $R = 1000$. They are used to calculate the tube controller gain K as the corresponding LQR gain and terminal penalty P to satisfy Assumption 13.

The simulation results are obtained from solving optimization problem (25) in a receding-horizon manner in MATLAB. The sets are obtained using the tools by Löfberg (2004), Herceg et al. (2013), and Riverso et al. (2013).

Fig. 4 shows the trajectories of plant and actuator states, as well as control inputs, starting at $x(t_0) = (-7.7 \;\; 5.7)^{\mathsf{T}}$ and $z(t_0) = 0$. Note that all constraints are satisfied and the plant is successfully stabilized at the origin. The resulting feasible region for the plant states is shown in Fig. 5.

6. SUMMARY AND OUTLOOK

We considered the problem of hierarchical control, where the upper-level controller does not have a detailed model of the lower level. We presented a model predictive control scheme that stabilizes a system consisting of known plant dynamics and an actuator, whose dynamics are unknown to the MPC controller. To achieve stability and constraint satisfaction, the controller and the actuator agree on

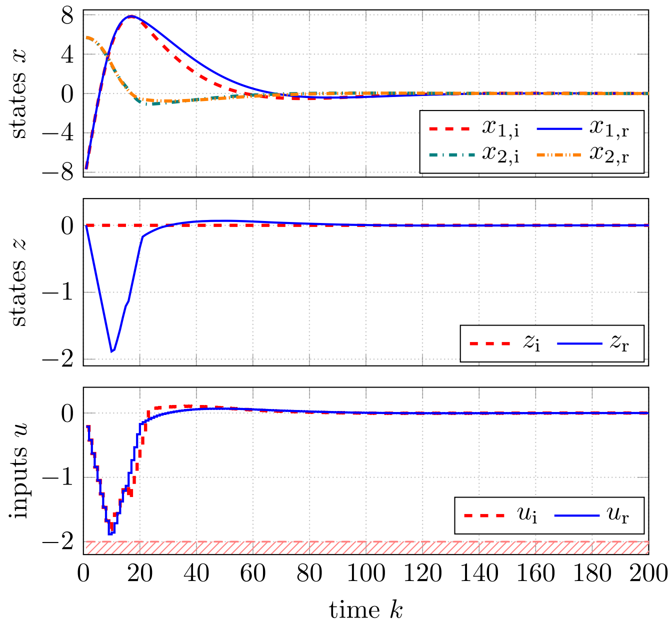


Fig. 4. Simulation results for the closed loop, consisting of the plant (33), the designed controller, and either the "ideal" actuator (subscript i) or the real actuator (34) (subscript r). Top to bottom: Evolution of the plant states, evolution of the actuator states, and control input u for the ideal system and the real system.

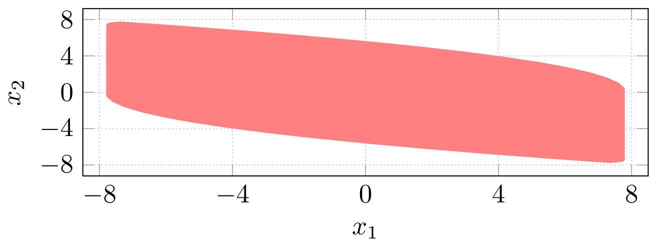


Fig. 5. Region of plant states x for which the optimization problem is feasible.

"contract" during the design phase. This contract consists of a bound on the error resulting from the unknown intermediate actuator. To achieve the desired accuracies, the actuator provides bounds on the allowable inputs and input changes. First, we derived a robust positive invariant set for the error bound, in a discrete-time setting. With this bound, we setup a tube-based MPC scheme to guarantee constraint satisfaction and stability. The simulation example provided further insights into the control scheme.

The approach allows to improve controller performance in settings where not all information at the cost level is available, e.g. due to privacy, legal, or modularity reasons. This is often the case in industrial applications, where different manufacturers might consider internal actuator dynamics as proprietary knowledge. Furthermore, the approach allows constructing modular controller-actuator-plant cascades, in which actuators can be exchanged—e.g. for other models or vendors with different system dynamics—as long as they satisfy the same accuracy contract.

Future work and extensions will consider nonlinear dynamics for plant and actuator as well as various descriptions of uncertainty.

REFERENCES

Blanchini, F. and Miani, S. (2015). *Set-Theoretic Methods in Control*. Systems & Control: Foundations & Applications. Birkhäuser, Basel, 2nd edition.

Boyd, S.P. and Vandenberghe, L. (2004). *Convex Optimization*. Cambridge University Press, Cambridge.

Bäthge, T., Lucia, S., and Findeisen, R. (2016). Exploiting Models of Different Granularity in Robust Predictive Control. In *Proc. 55th IEEE Conference on Decision and Control*, 2763–2768.

Campbell, M., Egerstedt, M., How, J.P., and Murray, R.M. (2010). Autonomous driving in urban environments: approaches, lessons and challenges. *Philosophical Transactions of the Royal Society A*, 368(1928), 4649–4672.

Di Cairano, S. and Borrelli, F. (2016). Reference tracking with guaranteed error bound for constrained linear systems. *IEEE Trans. Autom. Contr.*, 61(8), 2245–2250.

Herceg, M., Kvasnica, M., Jones, C.N., and Morari, M. (2013). Multi-Parametric Toolbox 3.0. In *Proc. 13th European Control Conference*, 502–510.

Kögel, M. and Findeisen, R. (2015). Robust output feedback model predictive control using reduced order models. In *Proc. 9th International Symposium on Advanced Control of Chemical Processes*, 1009–1015.

Kögel, M. and Findeisen, R. (2018). Improved Robust Decentralized MPC. In *Proc. 16th European Control Conference*, 312–318.

Lucia, S., Kögel, M., and Findeisen, R. (2015). Contract-based Predictive Control of Distributed Systems with Plug and Play Capabilities. In *Proc. 5th IFAC Conference on Nonlinear Model Predictive Control*, 205–111.

Lucia, S., Kögel, M., Zometa, P., Quevedo, D.E., and Findeisen, R. (2016). Predictive control, embedded cyberphysical systems and systems of systems – A perspective. *Annual Reviews in Control*, 41, 193–207.

Löfberg, J. (2004). YALMIP: A Toolbox for Modeling and Optimization in MATLAB. In *Proc. 13th IEEE Int. Symp. on Comp. Aided Control System Design*, 284–289.

Maciejowski, J.M. (2002). *Predictive Control with Constraints*. Prentice Hall, Harlow.

Mayne, D.Q. (2014). Model predictive control: Recent developments and future promise. *Automatica*, 50(12), 2967–2986.

Mayne, D.Q., Raković, S.V., Findeisen, R., and Allgöwer, F. (2006). Robust output feedback model predictive control of constrained linear systems. *Automatica*, 42(7), 1217–1222.

Mayne, D.Q., Seron, M.M., and Raković, S.V. (2005). Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41, 219–224.

Ogata, K. (1995). *Discrete-time Control Systems*. Prentice Hall, Englewood Cliffs, NJ, 2nd edition.

Rawlings, J.B., Mayne, D.Q., and Diehl, M.M. (2017). *Model Predictive Control: Theory, Computation, and Design*. Nob Hill Publishing, Madison, WI, 2nd edition.

Riverso, S., Battocchio, A., and Ferrari-Trecate, G. (2013). PnPMPC toolbox. URL `http://sisdin.unipv.it/pnpmpc/pnpmpc.php`.