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Robust Nonlinear Model Predictive Control with Constraint Satisfaction: A Relaxation-based Approach

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Abstract: A nonlinear model predictive control scheme guaranteeing robust constraint satisfaction is presented. The scheme is applicable to polynomial or rational systems and guarantees that state, terminal, and output constraints are robustly satisfied despite uncertain and bounded disturbances, parameters, and state measurements or estimates. In addition, for a suitably chosen terminal set, feasibility of the underlying optimization problem at a time instance guarantees that the constraints are robustly satisfied for all future time instances. The proposed scheme utilizes a semi-infinite optimization problem reformulated as a bilevel optimization problem: The outer program determines an input minimizing a performance index for a nominal nonlinear system, while several inner programs certify robust constraint satisfaction. We use convex relaxations to deal with the nonlinear dynamics in the inner programs efficiently. A simulation example is presented to demonstrate the approach.

Keywords: Model predictive control; nonlinear systems; uncertain systems.

1. INTRODUCTION

An important question about model predictive control (MPC) is its robustness to model uncertainty and noise, see e.g. (Bemporad and Morari, 1999). Of particular interest are robust stability, robust performance, and robust constraint satisfaction. Another issue is that recursive feasibility can be lost, i.e. the repeated solvability of the problem, cf. e.g. (Löfberg, 2012).

The problem of robustness in MPC has been considered extensively and approaches are spanning from robust optimization (e.g. (Kerrigan and Mayne, 2002; Houska and Diehl, 2012)), worst-case or minimax MPC (e.g. (Campo and Morari, 1987; Löfberg, 2003a,b)), to stochastic (e.g. (Mesbah et al., 2014; Cannon et al., 2011b) and references within) and scenario-based approaches (e.g. (Calafiore and Fagiano, 2013; Lucia and Engell, 2013)). A prominent approach is tube-MPC, used e.g. by Langson et al. (2004) and Raković et al. (2010, 2012) for linear systems, by Mayne and Kerrigan (2007) and Cannon et al. (2011a) for nonlinear systems, and by Cannon et al. (2011b) for systems subject to stochasticity. In tube-MPC, the bundle of perturbed trajectories is bounded by a tube around a *nominal* trajectory. Robustness can be achieved by enforcing the cross sections of the tubes to be robust, positively

invariant. Other approaches to robust MPC are based on game theory (Chen et al., 1997, 2007), LMIs (Kothare et al., 1996; Böhm et al., 2010) and reachable sets. Robust MPC of constrained discrete-time nonlinear systems based on the concept of reachable sets has for example been investigated by Bravo et al. (2006) using zonotopes and by Limon et al. (2005) using interval arithmetics. Robustness of MPC was also analyzed by e.g. Limon et al. (2009), Findeisen et al. (2011) and Grimm et al. (2004).

This paper presents a nonlinear MPC scheme for robust constraint satisfaction of systems with polynomial or rational dynamics subject to bounded uncertainties and disturbances. In particular, robust satisfaction of state and output constraints is guaranteed. The robust nonlinear MPC problem is formulated, similar to tube-MPC, by considering a *nominal* trajectory for performance and a bundle of trajectories for all possible uncertainty realizations for robust constraint satisfaction (see Sec. 2). The presented formulation results in an optimization problem with a bilevel structure. In the outer problem, a nominal model is used to determine an input sequence minimizing a performance index. To guarantee robust constraint satisfaction for an input sequence, a nonlinear model is used in the inner programs to predict the uncertainty propagation and to provide certificates that constraints are not violated (see Sec. 3). For the certificates, the nonlinear dynamics in the inner programs are relaxed. We propose to use a linear relaxation with tight bounds on bilinear and higher order monomials appearing in the system dynamics to provide a tight yet solvable relaxation. In Sec. 5, we analyze recursive

feasibility and the approach is illustrated for an example in Sec. 6. This work builds on a set-based analysis and fault diagnosis framework for nonlinear systems (Streif et al., 2013b; Paulson et al., 2014).

2. PROBLEM FORMULATION

Consider the following nonlinear, discrete-time system

$$\begin{aligned} x_{i+1} &= f(x_i, u_i, v_i, p) \\ y_i &= h(x_i, u_i, v_i, p), \end{aligned} \qquad (1)$$

where i denotes the time index. The time-invariant parameters are denoted by $p \in \mathbb{R}^{n_p}$. $x_i \in \mathbb{R}^{n_x}$, $u_i \in \mathbb{R}^{n_u}$, $v_i \in \mathbb{R}^{n_v}$, and $y_i \in \mathbb{R}^{n_y}$ denote the states, inputs, time-variant disturbances, and outputs, respectively. The functions f and h are assumed to be polynomial or rational functions.

We assume that the states, inputs, and outputs need to be restricted to convex, compact sets:

$$u_i \in \mathcal{U}, \qquad y_i \in \mathcal{Y}, \qquad x_i \in \mathcal{X}. \qquad (2)$$

The system dynamics (1) are subject to the following uncertainties: The time-invariant parameters p and time-variant disturbances v_i are assumed to take values from the convex and compact sets $\mathcal{P}$ and $\mathcal{V}$, respectively, and nominal values $p^{\mathrm{n}} \in \mathcal{P}, v^{\mathrm{n}} \in \mathcal{V}$ are available. Additionally, we allow imprecise measurements or estimates of the state x_i in form of a nominal estimate $\hat{x}_i$ and a set $\mathcal{X}_i^{\mathrm{m}}$ which contains $\hat{x}_i$ and the real state. The set $\mathcal{X}_i^{\mathrm{m}}$ is assumed to be convex and compact. Note that this formulation also allows considering exact state measurements by choosing $\mathcal{X}_i^{\mathrm{m}} = \{\hat{x}_i\}$.

We propose to control the constrained system (1), (2) using robust nonlinear model predictive control in order to robustly satisfy the constraints and *optimize* the nominal closed loop behavior. More precisely, starting at time instance k and using a control horizon N, we aim to determine for the current nominal state $\hat{x}_k \in \mathcal{X}_k^{\mathrm{m}}$ an open-loop input $\overline{u}_k, \ldots, \overline{u}_{k+N-1}$ such that the behavior of the nominal system (1) with nominal values p^{n} and v^{n} is optimized and that the constraints (2) are satisfied for all possible realizations of the system subject to the uncertainties. This is denoted as *robust constraint satisfaction* which means, in mathematical terms, that for the given input sequence $\overline{u}_k, \ldots, \overline{u}_{k+N-1}$ and for all $\widetilde{x}_k \in \mathcal{X}_k^{\mathrm{m}}$, $\widetilde{p} \in \mathcal{P}$, $\widetilde{v}_k \in \mathcal{V}, \ldots, \widetilde{v}_{k+N-1} \in \mathcal{V}$, the predicted state trajectory $\widetilde{x}_k, \ldots, \widetilde{x}_{k+N-1}$ and predicted output sequence $\widetilde{y}_k, \ldots, \widetilde{y}_{k+N-1}$ satisfy the constraints (2), and that the predicted terminal state $\widetilde{x}_{k+N}$ satisfies a possibly present terminal constraint in $\mathcal{X}^{\mathrm{f}} \subseteq \mathcal{X}$, see Sec. 5.

The resulting control task can be formally stated as:

Problem 1 (Robust nonlinear MPC): *Consider at each time k the following nonlinear optimal control problem with quadratic performance index:*

$$\underset{\overline{u}_k, \ldots, \overline{u}_{k+N-1}}{\text{minimize}} \sum_{i=k}^{k+N} \overline{x}_i^{\mathsf{T}} Q_i \overline{x}_i + \sum_{i=k}^{k+N-1} \overline{u}_i^{\mathsf{T}} R_i \overline{u}_i \quad , \qquad (3a)$$

subject to:

$$\overline{x}_{i+1} = f(\overline{x}_i, \overline{u}_i, v^{\mathrm{n}}, p^{\mathrm{n}}) \qquad (3b)$$
$$\overline{x}_k = \hat{x}_k \qquad (3c)$$
$$\overline{u}_i \in \mathcal{U} \qquad (3d)$$

and $\forall \widetilde{x}_k \in \mathcal{X}_k^{\mathrm{m}}, \forall \widetilde{p} \in \mathcal{P}, \forall \widetilde{v}_i \in \mathcal{V}, i = k, \ldots, k+N-1$:

$$\widetilde{x}_{i+1} = f(\widetilde{x}_i, \overline{u}_i, \widetilde{v}_i, \widetilde{p}) \qquad (3e)$$
$$\widetilde{y}_i = h(\widetilde{x}_i, \overline{u}_i, \widetilde{v}_i, \widetilde{p}) \qquad (3f)$$
$$\widetilde{x}_i \in \mathcal{X} \qquad (3g)$$
$$\widetilde{x}_{k+N} \in \mathcal{X}^{\mathrm{f}} \qquad (3h)$$
$$\widetilde{y}_i \in \mathcal{Y}, \qquad (3i)$$

where Q_i and R_i are symmetric, positive definite weighting matrices. The solution of (3) provides the input $\overline{u}_k$, which is used as feedback $u_k = \overline{u}_k$ for the system (1).

Note that the optimization problem (3) depends on the nominal state estimate $\hat{x}_k$, set estimate $\mathcal{X}_k^{\mathrm{m}}$, and the nominal values of the disturbances and parameters, $v^{\mathrm{n}} \in \mathcal{V}$ and $p^{\mathrm{n}} \in \mathcal{P}$, respectively. The optimal input is determined based on the nominal system (3b). The system (3e)–(3i) is used to predict (for all possible uncertainties) the state trajectories and to certify that all constraints are satisfied.

Remark 1 (Choice of nominal system): *We assume that the nominal system is the nonlinear system (1) with known nominal parameters and uncertainties. Instead and as discussed in Sec. 4, one can also consider the linearization of the nonlinear system or any other system preferred for computational reasons.*

Problem (3) is a nonlinear semi-infinite programming problem due to the constraints (3e)–(3i) that are required to hold for all uncertainties $\mathcal{P}$, $\mathcal{V}$, and $\mathcal{X}_k^{\mathrm{m}}$. This type of problem is challenging to solve (see e.g. (Bard, 1998; Löfberg, 2003b)). The solution approach in this work is the reformulation as a bilevel program: the outer program (corresponding to Eqs. (3b)–(3d)) takes care of the optimization of the performance specification, while the inner program (corresponding to Eqs. (3a)–(3i)) provides certificates for robust constraint satisfaction for all trajectories realizable due to the uncertainties. To efficiently compute the robustness certificates while not losing guarantees, we relax the nonlinear inner program using a convex relaxation framework that leads to a linear program (a similar approach has been proposed in (Streif et al., 2013a; Paulson et al., 2014)). The following sections elaborate on these ideas.

3. ROBUST CONSTRAINT SATISFACTION OF INPUT SEQUENCES

This section assumes input sequences $\overline{u}_k, \ldots, \overline{u}_{k+N-1}$ to be given and aims to derive certificates for the robust satisfaction of the state, output, and terminal constraints despite the uncertainties $\mathcal{P}$ and $\mathcal{V}$. This is achieved by a reformulation of the constraints for the robustness certificates as illustrated in Fig. 1, and by using convex relaxations that allow an efficient computation. Based on these certificates, the main results will be presented in Sec. 4, namely the robust nonlinear MPC scheme.

In many cases, state, output, and terminal constraints can be expressed or approximated by convex, compact sets of the following form:
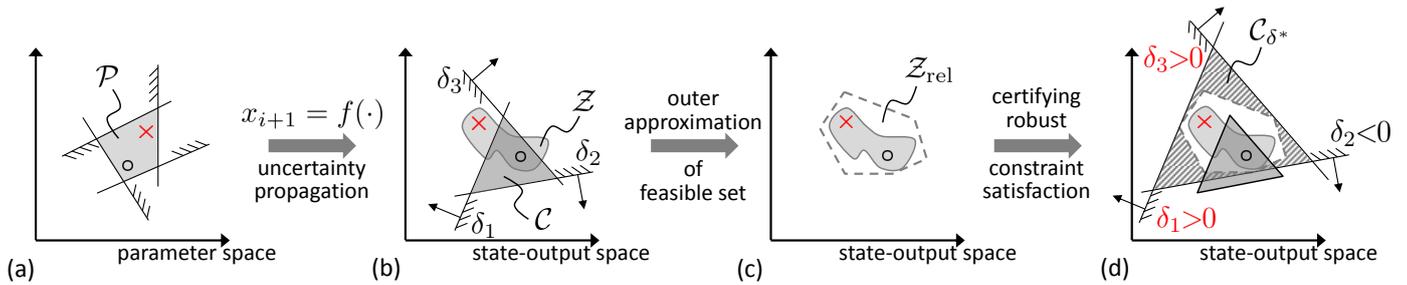
Fig. 1. Illustration of the approach in case of parametric uncertainties $\mathcal{P}$. Parametric uncertainties (light gray area $\mathcal{P}$) in (a) are propagated through the nonlinear dynamics onto the feasible set (light gray area $\mathcal{Z}$) in (b). Certain parameter combinations may lead to satisfaction (o) or violation ($\times$) of the constraint set $\mathcal{C}$. To check whether constraint-violating parameters exist, first (c), the (possibly) non-convex feasible set is convexly outer-approximated as indicated (white area with dashed outline $\mathcal{Z}_{\mathrm{rel}}$); second (d), each facet of the constraint set $\mathcal{C}$ is translated by the corresponding δ_j, $j = 1, 2, 3$, until the resulting set $\mathcal{C}_{\delta*}$ fully contains the (outer-approximated) feasible set $\mathcal{Z}_{\mathrm{rel}}$. The arrows in (b) and (d) point into the increasing direction of δ_j, and increasing δ_j move the half-space into the direction of the arrow, thus relaxing the constraint. Depending on the resulting δ_js, the following can be concluded: If all $\delta_j^* \leq 0$, then $\mathcal{Z} \subseteq \mathcal{Z}_{\mathrm{rel}} \subseteq \mathcal{C}_{\delta*} \subseteq \mathcal{C}$ and *no* constraint-violating parameters exist.

Assumption 1 (State, output, and terminal constraints): *State constraints $\mathcal{X}$, output constraints $\mathcal{Y}$, and terminal constraints $\mathcal{X}^{\mathrm{f}}$ can be represented by a convex, compact set constituted by n_{C} half-spaces*

$$\mathcal{C} := \left\{ \zeta_{xy} : c_j^{\mathsf{T}} \zeta_{xy} \leq 0, \ \forall j \in \mathcal{J}_{\mathrm{C}} \right\}, \tag{4}$$

where $\zeta_{xy} := \left[1, \widetilde{x}_k^{\mathsf{T}}, \ldots, \widetilde{x}_{k+N}^{\mathsf{T}}, \widetilde{y}_k^{\mathsf{T}}, \ldots, \widetilde{y}_{k+N-1}^{\mathsf{T}}\right]^{\mathsf{T}} \in \mathbb{R}^{n_{xy}}$, $c_j \in \mathbb{R}^{n_{xy}}$ and $\mathcal{J}_{\mathrm{C}} := \{1, \ldots, n_{\mathrm{C}}\}$. We assume that the constraint set $\mathcal{C}$ is not empty.

What has to be shown is that all state and output values reachable by the system (for all possible uncertainties) are contained in the constraint set $\mathcal{C}$. This is addressed next.

3.1 Robust Constraint Satisfaction

For the subsequent purposes, we define the feasible (or reachable) set $\mathcal{Z}$:

Definition 1 (Feasible set): *The feasible set $\mathcal{Z}$ of system (1) is the set of points (in the state-output space) that can be reached for all possible uncertainties (i.e. $\mathcal{X}_k^{\mathrm{m}}$, $\mathcal{P}$, and $\mathcal{V}$) and for the given inputs $\overline{u}_i$ on the time interval $k, \ldots, k+N-1$. It is given by:*

$$\mathcal{Z} := \Big\{ \ \widetilde{x}_k^{\mathsf{T}}, \ldots, \widetilde{x}_{k+N}^{\mathsf{T}}, \widetilde{y}_k^{\mathsf{T}}, \ldots, \widetilde{y}_{k+N-1}^{\mathsf{T}} :$$
$$\widetilde{x}_{i+1} = f(\widetilde{x}_i, \overline{u}_i, \widetilde{v}_i, \widetilde{p}), \quad \forall i = k, \ldots, k+N-1,$$
$$\widetilde{y}_i = h(\widetilde{x}_i, \overline{u}_i, \widetilde{v}_i, \widetilde{p}), \quad \forall i = k, \ldots, k+N-1,$$
$$\forall \widetilde{x}_k \in \mathcal{X}_k^{\mathrm{m}}, \forall \widetilde{p} \in \mathcal{P}, \forall \widetilde{v}_i \in \mathcal{V}, \ \forall i = k, \ldots, k+N-1$$
$$\Big\} \subseteq \mathbb{R}^{n_{xy}-1}.$$

Furthermore, a convex outer approximation of $\mathcal{Z}$ is denoted by $\mathcal{Z}_{\mathrm{rel}}$, thus $\mathcal{Z} \subseteq \mathcal{Z}_{\mathrm{rel}}$.

In the following, the core idea is to check for each constraint in (4) whether it is satisfied or not. To this end, we outer-approximate the feasible set $\mathcal{Z}$ by a set $\mathcal{C}_{\delta*}$. The latter set is obtained by reformulating the right-hand side of each constraint in (4) to obtain $c_j^{\mathsf{T}} \zeta_{xy} \leq \delta_j^*$. Depending on the values of δ_j^*, one can then conclude whether $\mathcal{Z}$ is fully contained in $\mathcal{C}$ or not. If $\mathcal{Z} \subseteq \mathcal{C}$, $\nexists p \in \mathcal{P}$ and $\nexists v_k \in \mathcal{V}, \ldots, \nexists v_{k+N-1} \in \mathcal{V}$ and $\nexists \widetilde{x}_k \in \mathcal{X}_k^{\mathrm{m}}$ for which the system violates a constraint from (4). This idea is stated in the following Lemma:

Lemma 1 (Robust constraint satisfaction): *Assume the input sequence $\overline{u}_k, \ldots, \overline{u}_{k+N-1}$ and the feasible set $\mathcal{Z}$ to be given. Furthermore, assume $\forall j \in \mathcal{J}_{\mathrm{C}}$:*

$$\delta_j^* := \underset{\zeta_{xy}, \delta_j}{\text{maximize}} \quad \delta_j$$
$$\text{subject to:} \quad \begin{aligned} \zeta_{xy} &\in \mathcal{Z} \\ c_j^{\mathsf{T}} \zeta_{xy} &= \delta_j. \end{aligned}$$

Iff $\delta_j^* \leq 0 \ \forall j \in \mathcal{J}_{\mathrm{C}}$, then $\mathcal{Z} \subseteq \mathcal{C}$.

Proof: *Let us define the set*
$$\mathcal{C}_{\delta*} := \left\{ \zeta_{xy} : c_j^{\mathsf{T}} \zeta_{xy} \leq \delta_j^*, \ \forall j \in \mathcal{J}_{\mathrm{C}} \right\}.$$
It trivially follows from the definitions of $\mathcal{C}$ and $\mathcal{C}_{\delta*}$ that $\mathcal{C}_{\delta*} \subseteq \mathcal{C}$, iff $\delta_j^* \leq 0, \ \forall j \in \mathcal{J}_{\mathrm{C}}$. Now assume $z \in \mathcal{Z}$ and $z \notin \mathcal{C}_{\delta*}$. The latter two statements hold, iff $\exists \hat{\delta}_j > \delta_j^*$ with $c_j^{\mathsf{T}} \begin{bmatrix} 1 \\ z \end{bmatrix} = \hat{\delta}_j$. However, this is a contradiction due to the maximization of δ_j. It follows that, iff $\delta_j^* \leq 0, \ \forall j \in \mathcal{J}_{\mathrm{C}}$, $\mathcal{Z} \subseteq \mathcal{C}_{\delta*}$; consequently $\mathcal{Z} \subseteq \mathcal{C}_{\delta*} \subseteq \mathcal{C}$ and $\mathcal{Z} \subseteq \mathcal{C}$. ∎

In general it is difficult to determine the δ_js owing to the non-convexity of the set $\mathcal{Z}$. We therefore use relaxations to derive a convex outer approximation $\mathcal{Z}_{\mathrm{rel}}$ of the set $\mathcal{Z}$, which we then use in Lemma 1. With that, it trivially follows that $\mathcal{Z} \subseteq \mathcal{Z}_{\mathrm{rel}} \subseteq C$, iff $\delta_j^* \leq 0 \ \forall j \in \mathcal{J}_{\mathrm{C}}$, so that we can guarantee robust constraint satisfaction while being computationally tractable. The approach is illustrated in Fig. 1.

3.2 Relaxation-based Certificate

Define the vector $\zeta := \left[\zeta_{xy}^{\mathsf{T}}, \widetilde{v}_k^{\mathsf{T}}, \ldots, \widetilde{v}_{k+N-1}^{\mathsf{T}}, \widetilde{p}^{\mathsf{T}}, \mathrm{h.o.m.}\right]^{\mathsf{T}} \in \mathbb{R}^{n_\zeta}$, which contains the constant 1 and all variables appearing in ζ_{xy} and in Eqs. (3e)–(3f), as well as all higher order monomials (h.o.m.) needed to represent all polynomial equations in a quadratic form. Note that the choice of ζ might not be uniquely determined, but we do not elaborate on this here. Also note that the input is assumed constant (since it is given by the outer program) and therefore only affects the coefficients of the matrices of the following optimization problem.

To determine δ_j for the jth constraint in the set $\mathcal{C}_{\delta*}$ (see Lemma 1), we can write:

$$\begin{aligned}
\underset{\zeta, \delta_j}{\text{maximize}} \quad & \delta_j \\
\text{subject to:} \quad & \zeta^\mathsf{T} \widetilde{A}(\overline{u})_i \zeta = 0, \ \forall i \in \mathcal{J}_\text{eq} \\
& \zeta^\mathsf{T} \widetilde{B}(\overline{u})_i \zeta \leq 0, \ \forall i \in \mathcal{J}_\text{ineq} \\
& \begin{bmatrix} c_j^\mathsf{T} & \mathbf{0} \end{bmatrix} \zeta = \delta_j.
\end{aligned} \tag{5}$$

Here, $\widetilde{A}(\overline{u})_i$ and $\widetilde{B}(\overline{u})_i$ are input-dependent, symmetric matrices accounting for the dynamics (3e), output maps (3f), uncertain initial estimates or measurements $\widetilde{x}_k$, and sets $\mathcal{P}$ and $\mathcal{V}$. The matrices $\widetilde{A}(\overline{u})_i$ and $\widetilde{B}(\overline{u})_i$ are obtained by a quadratic reformulation employing ζ (see (Streif et al., 2013a) for further details). $\mathcal{J}_\text{eq} := \{1, \ldots, n_\text{eq}\}$ and $\mathcal{J}_\text{ineq} := \{1, \ldots, n_\text{ineq}\}$ are index sets with n_eq and n_ineq being the number of equalities and inequalities, respectively. The row vector of zeros $\mathbf{0}$ has length $n_\zeta - n_{xy}$.

We choose the following linear relaxation of the inner program (details see (Streif et al., 2013a)):

$$\begin{aligned}
\underset{Z, \delta_j}{\text{maximize}} \quad & \delta_j \\
\text{subject to:} \quad & \text{trace}\big(\widetilde{A}(\overline{u})_i Z\big) = 0, \ \forall i \in \mathcal{J}_\text{eq} \\
& \text{trace}\big(\widetilde{B}(\overline{u})_i Z\big) \leq 0, \ \forall i \in \mathcal{J}_\text{ineq} \\
& \text{trace}\big(\widetilde{D}(\overline{u})_i Z\big) \leq 0, \ \forall i \in \mathcal{J}_\text{relax} \\
& \begin{bmatrix} c_j^\mathsf{T} & \mathbf{0} \end{bmatrix} Z e = \delta_j.
\end{aligned} \tag{6}$$

Here, $Z \in \mathbb{R}^{n_\zeta \times n_\zeta}$ is a symmetric matrix, $e := [1, 0, \ldots, 0]^\mathsf{T} \in \mathbb{R}^{n_\zeta}$ is a unit vector. The relaxation is obtained by introducing the symmetric and rank-1 matrix variable $Z := \zeta \zeta^\mathsf{T}$, by rewriting all constraints in terms of Z, and by dropping the rank-1 constraint on Z. In addition to the constraints in (5), so-called tightening constraints (accounted for by the matrices $\widetilde{D}(\overline{u})_i$) are introduced. These constraints usually correspond to the McCormick constraints on bilinear and higher order monomials in Z and are important to reduce the relaxation error (Streif et al., 2013a).

The resulting linear optimization problem (6) can be solved efficiently to derive the δ_j^* in Lemma 1. Thus, Lemma 1, together with the convex relaxations, provides certificates for robust constraint satisfaction for given input sequences. This result will be used next to determine an optimal input under robust feasibility, thus solving Problem 1.

4. DETERMINING SUBOPTIMAL FEASIBLE INPUTS

In this section, we restate the robust nonlinear MPC problem 1 as a nonlinear bilevel programming problem. The idea is the following: The outer program deals with Eqs. (3b)–(3d) and provides an input sequence $\overline{u}_k, \ldots, \overline{u}_{k+N-1}$ to several inner programs which provide certificates for robust feasibility in terms of the δ_js (cf. Lemma 1). The δ_js are restricted to non-positive values by the outer problem.

To unify the notation, the outer program can be reformulated in a similar manner as the inner program (5) using the vector $\xi := \big[1, \overline{x}_k^\mathsf{T}, \ldots, \overline{x}_{k+N}^\mathsf{T}, \overline{y}_k^\mathsf{T}, \ldots, \overline{y}_{k+N-1}^\mathsf{T}, \overline{u}_k^\mathsf{T}, \ldots, \overline{u}_{k+N-1}^\mathsf{T}, \text{h.o.m.}\big]^\mathsf{T}$. Note that ξ in contrast to ζ contains the input variables

as decision variables, but not the parameters and disturbances, which are treated as constant and accounted for in the coefficients of the matrices in:

$$\begin{aligned}
\underset{\xi}{\text{minimize}} \quad & \xi^\mathsf{T} J \xi \\
\text{subject to:} \quad & \xi^\mathsf{T} \overline{A}(p^\text{n}, v^\text{n})_i \xi = 0, \forall i \in \overline{\mathcal{J}}_\text{eq} \\
& \xi^\mathsf{T} \overline{B}(p^\text{n}, v^\text{n})_i \xi \leq 0, \forall i \in \overline{\mathcal{J}}_\text{ineq}.
\end{aligned} \tag{7}$$

Here, $\overline{A}(p^\text{n}, v^\text{n})_i$ and $\overline{B}(p^\text{n}, v^\text{n})_i$ are symmetric matrices accounting for the dynamics (3b), consistency with the nominal state $\hat{x}_k$ (3c), and input constraints (3d). $\overline{\mathcal{J}}_\text{eq} := \{1, \ldots, \overline{n}_\text{eq}\}$ and $\overline{\mathcal{J}}_\text{ineq} := \{1, \ldots, \overline{n}_\text{ineq}\}$ are index sets with $\overline{n}_\text{eq}$ and $\overline{n}_\text{ineq}$ being the number of equalities and inequalities, respectively. The matrix J in the objective is a symmetric, positive definite matrix representing the quadratic objective (3a) in Problem 1.

Using Lemma 1, (6), and (7), the following main result can be stated:

Theorem 1 (Suboptimal robust nonlinear MPC):
The solution $\overline{u}_k^*, \ldots, \overline{u}_{k+N-1}^*$ of the optimization problem

$$\begin{aligned}
\underset{\xi}{\text{minimize}} \quad & \xi^\mathsf{T} J \xi \\
\text{subject to:} \quad & \xi^\mathsf{T} \overline{A}(p^\text{n}, v^\text{n})_i \xi = 0, \forall i \in \overline{\mathcal{J}}_\text{eq} \\
& \xi^\mathsf{T} \overline{B}(p^\text{n}, v^\text{n})_i \xi \leq 0, \forall i \in \overline{\mathcal{J}}_\text{ineq} \\
& \delta_j^* \leq 0, \forall j \in \mathcal{J}_\text{C} \\
& \left(\begin{aligned}
\delta_j^* := \underset{Z, \delta_j}{\text{maximize}} \ & \delta_j \\
\text{subject to:} \ & \text{trace}\big(\widetilde{A}(\overline{u})_i Z\big) = 0, \forall i \in \mathcal{J}_\text{eq} \\
& \text{trace}\big(\widetilde{B}(\overline{u})_i Z\big) \leq 0, \forall i \in \mathcal{J}_\text{ineq} \\
& \text{trace}\big(\widetilde{D}(\overline{u})_i Z\big) \leq 0, \forall i \in \mathcal{J}_\text{relax} \\
& \begin{bmatrix} c_j^\mathsf{T} & \mathbf{0} \end{bmatrix} Z e = \delta_j
\end{aligned} \right)_{j=1}^{n_\text{C}}
\end{aligned} \tag{8}$$

provides a suboptimal, robustly feasible input sequence for which it is guaranteed that the state $\mathcal{X}$, output $\mathcal{Y}$, input $\mathcal{U}$, and terminal constraints $\mathcal{X}^\text{f}$ are satisfied $\forall p \in \mathcal{P}$ and $\forall v_k \in \mathcal{V}, \ldots, \forall v_{k+N-1} \in \mathcal{V}$.

Proof: *Feasibility, in particular for input constraints, is guaranteed due to the nonlinear solution strategy of the outer program. Enforcing $\delta_j^* \leq 0 \ \forall j \in \mathcal{J}_\text{C}$, as provided by maximization of the δ_js in the inner programs, forces the optimization to choose an input that guarantees robust feasibility by virtue of Lemma 1.* ∎

Remark 2 (Solution of the bilevel optimization problem): *Note that the optimization problem in Theorem 1 is of a bilevel structure with a nonlinear outer problem and convex inner problems. Nonlinear solvers can deal with the inner convex (linear) program by explicitly calling e.g. CPLEX. The decision variables δ_j of the inner problem provide measures of robustness and can be used by the nonlinear program solver to find a locally optimal solution.*

Also notice that the optimization problem is in general non-smooth, because the mapping from Z to the optimal δ_js might not be differentiable everywhere. Therefore, utilizing smooth nonlinear solvers for the outer problem might be challenging. An alternative is to use non-smooth optimization techniques, cf. (Clarke, 1990; Mäkelä, 2002).

Remark 3 (Simplifications and convexification of the outer program): *A significant amount of complexity is due to the nonlinear constraints in the outer problem of Theorem 1. These constraints are required in order to account for the dependency of the performance index on the states. The problem simplifies significantly, if only optimality of the input is considered. Another possibility to reduce complexity is by approximating the nonlinear dynamics in the outer loop by a linear one; this then yields a linear outer problem which can be solved efficiently. Robust feasibility with respect to the nonlinear dynamics is still guaranteed due to the inner program. Note that the overall bilevel optimization problem would still not be convex, even if the outer program were convex (see e.g. (Bard, 1998)).*

Another computational simplification would be a convexification of the outer program (7). However, nonlinear feasibility in the outer program is then no longer guaranteed, but the inner program still guarantees robust feasibility.

In the next section, we analyze recursive feasibility of the suboptimal robust nonlinear MPC scheme and demonstrate the suboptimal robust nonlinear MPC scheme in Sec. 6 considering a simple example system.

5. RECURSIVE FEASIBILITY

In the following, we consider exact state measurements, i.e. that $\hat{x}_k = x_k$, $\mathcal{X}_k^{\mathrm{m}} = \{x_k\}$. We show that the presented scheme is ultimately recursively feasible, which means that if the optimization problem (8) is feasible, then it will be feasible again after at least N steps:

Definition 2 (Ultimate recursive feasibility): *A nonlinear MPC scheme with horizon N is called ultimately recursively feasible, if feasibility at time instance k guarantees feasibility after at least N time instances for any disturbance realizations $v_k \in \mathcal{V}, \ldots, v_{k+N-1} \in \mathcal{V}$ and uncertain parameters $p \in \mathcal{P}$.*

Note that with the concept of ultimate recursive feasibility, it is not guaranteed that (8) is feasible at the time instances $k+1, k+2, \ldots, k+N-1$, if (8) is feasible at time instance k. In this case, since the problem (8) has no solution, it can not provide the optimal inputs required for the feedback. One remedy is to use the solution from time instance k, denoted by $\overline{u}_{k|k}^*, \ldots, \overline{u}_{k+N-1|k}^*$, as a backup by utilizing $u_{k+i} = \overline{u}_{k+i|k}^*$. Due to the robust feasibility, this guarantees that all constraints will be robustly satisfied, including the terminal constraints. In combination with ultimate recursive feasibility, this will guarantee that feasibility of (8) at time instance k guarantees feasibility for any further time instance.

In order to guarantee ultimate recursive feasibility, we make the following robust positive invariance assumption on the terminal set $\mathcal{X}^{\mathrm{f}}$.

Assumption 2 (Terminal set, terminal control law): *The terminal set $\mathcal{X}^{\mathrm{f}} \subseteq \mathcal{X}$ and terminal control laws $\kappa_i(\overline{x})$, $i = 0, \ldots, N-1$, are such that $\kappa_i(\overline{x}) \in \mathcal{U}$. Moreover, for any $\overline{x} \in \mathcal{X}^{\mathrm{f}}$, $p \in \mathcal{P}$, and $v_i \in \mathcal{V}$, $i = 0, \ldots, N-1$,*

$\widetilde{x}_{i+1}^{\mathrm{rel}} \in f^{\mathrm{rel}}(\widetilde{x}_i^{\mathrm{rel}}, \kappa_i(\overline{x}), v_i, p)$, $\widetilde{y}_i^{\mathrm{rel}} \in h^{\mathrm{rel}}(\widetilde{x}_i^{\mathrm{rel}}, \kappa_i(\overline{x}), v_i, p)$, *where $\widetilde{x}_0^{\mathrm{rel}} = \overline{x}$, we have $\widetilde{y}_i^{\mathrm{rel}} \in \mathcal{Y}$ and $\widetilde{x}_{k+N}^{\mathrm{rel}} \in \mathcal{X}^{\mathrm{f}}$.*

The relaxed dynamics and output maps are denoted by f^{rel} and h^{rel}, respectively. Notice that we require that the terminal constraint is robustly positive invariant with respect to the relaxation (and consequently also the nominal system) over N steps. Determining such a terminal set might be in general challenging. However, for a possible candidate set, we can verify the above assumptions based on relaxations.

With the above assumptions, we obtain:

Proposition 1 (Ultimate recursive feasibility): *Let Assumption 2 hold and assume that the state x_k is exactly available. Then, the robust nonlinear MPC scheme (8) is ultimately recursively feasible.*

Proof: *We need to show that if (8) is feasible at time instance k, then (8) is feasible at time instance $k+N$ or earlier. Consequently, feasibility needs to be guaranteed for the case that (8) is feasible at k, but not necessarily at any time instances $k+1, \ldots, k+N-1$.*

Since (8) is feasible, $x_{k+N} \in \mathcal{X}^{\mathrm{f}}$. Consider the choice $\overline{u}_{k+N+i} = \kappa_i(x_{k+N})$, $\overline{x}_{k+N+i+1} = f(\overline{x}_{k+N+i}, \overline{u}_{k+N+i}, v^n, p^n)$, $i = 0, \ldots, N-1$, where $x_{k+N} = \overline{x}_{k+N}$. Moreover, since $x_{k+N} \in \mathcal{X}_f$, it is guaranteed that $\overline{u}_{k+N+i} \in \mathcal{U}$, compare Assumption 2, i.e. (3d) holds. Further, Assumption 2 guarantees that for any parameter uncertainty, disturbances, and for $\widetilde{x}_{k+N+i+1}^{\mathrm{rel}} \in f^{\mathrm{rel}}(\widetilde{x}_{k+N+i}, \kappa_i(x_{k+N}), \widetilde{v}_{i+k+N}, \widetilde{p})$, $\widetilde{y}_{k+N+i}^{r} \in h_r(\widetilde{x}_{k+N}, \kappa_i(x_{k+N}), v, p)$ with $\widetilde{x}_{k+N} = x_{k+N}$, we have $\widetilde{x}_{k+N+1}^{\mathrm{rel}} \in \mathcal{X}^{\mathrm{f}}$ and $\widetilde{y}_{k+N+1}^{\mathrm{rel}} \in \mathcal{Y}$. ∎

Remark 4 (Why only ultimate recursive feasibility?): *The above proof relies on the fact that we can evaluate the terminal control laws κ_i at time $k + N$ at a single point x_{k+N}, resulting in an input sequence $u_{k+N}, \ldots, u_{k+2N-1}$, since x_{k+N} will be known exactly.*

However, at the time instance $k + 1$, the future state x_{k+N} is still uncertain. Thus, the terminal control laws can not be evaluated at a single point (x_{k+N}). In order to establish recursive feasibility, one would need to choose a single input sequence for all possible states x_{k+N}, which guarantees certain conditions such as positive invariance of the terminal region. This seems to be only possible for special cases, e.g. asymptotic stable systems with $u = \kappa(x) = 0$.

6. EXAMPLE

Consider the two-dimensional system

$$x_{1,k+1} = x_{1,k} + 0.1\Big(x_{2,k} + \big(p + (1-p)x_{1,k}\big)u_k + v_{1,k}\Big)$$
$$x_{2,k+1} = x_{2,k} + 0.1\Big(x_{1,k} + \big(p - 4(1-p)x_{2,k}\big)u_k + v_{2,k}\Big),$$
$$(9)$$

which is a discretized version of the example in (Chen and Allgöwer, 1998) with additional disturbances $v_{1,k}$ and $v_{2,k}$.

The states $x_{1,k}$ and $x_{2,k}$ are constrained to $[-2, 2]$ and the input u_k to $[-2, 2]$. The uncertain parameter p and time-varying disturbances $v_{1,k}$, $v_{2,k}$ are assumed to be uncertain, but bounded to $[0.45, 0.55]$, $[-0.05, 0.05]$, and $[-0.05, 0.05]$, respectively. The state estimate error is bounded by $[-0.05, -0.05]^\mathsf{T} \leq x_k - \bar{x}_k \leq [0.05, 0.05]^\mathsf{T}$. The proposed predictive control scheme uses $p^\mathrm{n} = 0.5$ and $v_1^\mathrm{n} = v_2^\mathrm{n} = 0$ as nominal values. Moreover, we utilize a control horizon of $N = 10$ with a time step $\Delta t = 0.1$ and the weighting matrices are chosen as $Q_i = I$ and $R_i = 0.01$.

Fig. 2 presents the proposed scheme for a fixed initial state and different disturbance realizations. In detail, the following cases were simulated

   a) Nominal case: Correct parameter ($p = p^\mathrm{n}$), no estimation error or disturbances ($x_k = \bar{x}_k$, $v_{1,k} = v_{2,k} = 0$)
   b) Erroneous parameter ($p = 0.45$), no estimation error or disturbances ($x_k = \bar{x}_k$, $v_{1,k} = v_{2,k} = 0$)
   c) Erroneous parameter ($p = 0.55$), no estimation error or disturbances ($x_k = \bar{x}_k$, $v_{1,k} = v_{2,k} = 0$)
   d) Correct parameter ($p = p^\mathrm{n}$), no estimation error ($x_k = \bar{x}_k$), random disturbance $v_{1,k}, v_{2,k} \in [-0.05, 0.05]$
   e) Correct parameter ($p = p^\mathrm{n}$), uniformly distributed estimation error ($[-0.05, -0.05]^\mathsf{T} \leq x_k - \bar{x}_k \leq [0.05, 0.05]^\mathsf{T}$), uniformly distributed random disturbance $v_{1,k}, v_{2,k} \in [-0.05, 0.05]$

From the simulation, we observe that state and input constraints are satisfied. Moreover, without disturbances or state estimation errors (cases a)–c)), the scheme can bring the system to the origin from this initial value.

7. CONCLUSION

We presented a nonlinear MPC scheme that guarantees robust feasibility, exploiting a bilevel problem formulation and relaxation. The proposed algorithm is robust by construction and by employing relaxations. However, as typical for robust MPC approaches, the proposed scheme is conservative, first due to the requirement of robust feasibility for all disturbance realizations, second due to the relaxation of the nonlinear dynamics and outer approximation of the reachable sets. To reduce conservatism, a trade-off can be made between constraints that require guaranteed constraint satisfaction (such as safety critical bounds) and soft constraints. Besides that, the approach is computationally demanding, particularly due to the nonlinear dynamics in the outer program. Speed improvements can be achieved by simplifying or linearizing the dynamics in the outer program (see Remark 3), by parallelization of the n_C inner programs, or by using more sophisticated nonlinear numerical methods.

Note that only the generation of open-loop inputs is considered. To deal with the uncertainty during the prediction, often a combination of closed and open-loop feedbacks is employed, cf. (Raković et al., 2010, 2012). The presented approach can be expanded to such a use of closed and open-loop input combinations.
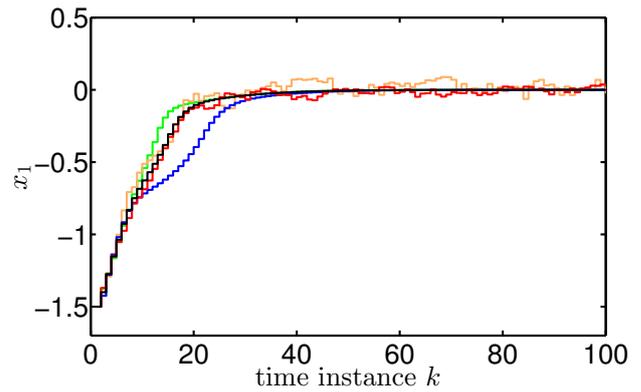

(a) Evolution of the state x_1
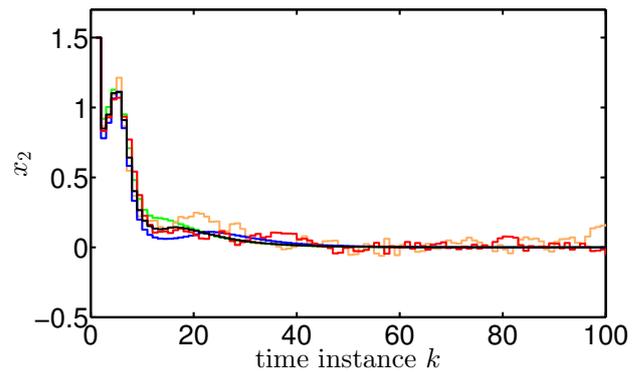
(b) Evolution of the state x_2
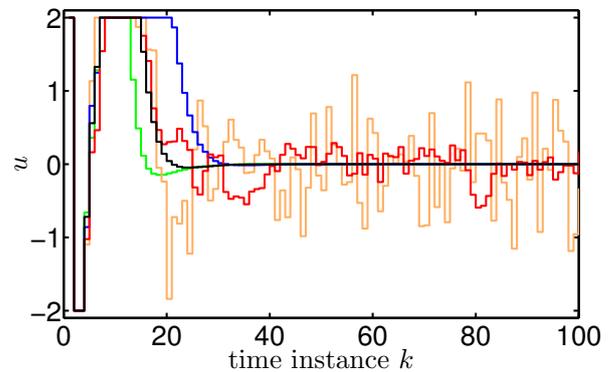
(c) Input signal u
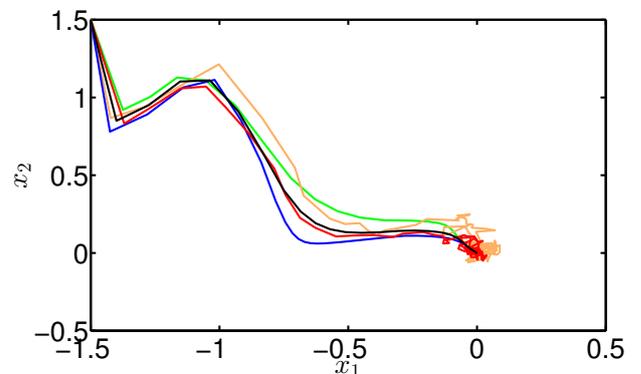
(d) Phase portrait

Fig. 2. Simulation results for the example system, for the initial state $x_0 = [-1.5, 1.5]^\mathsf{T}$. Blue: case a), black: case b), green: case c), red: case d), orange: case e).

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