Set Membership Parameter Estimation and Design of Experiments Using Homothety ⋆

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Abstract: In this note we address the problems of obtaining guaranteed and as good as possible estimates of system parameters for linear discrete–time systems subject to bounded disturbances. Some existing results relevant for the set–membership parameter identification and outer–bounding are first reviewed. Then, a novel method for characterizing the consistent parameter set based on homothety is offered; the proposed method allows for the utilization of general compact and convex sets for outer–bounding. Based on these results, we consider the one–step input design and identifiability problems in set–membership setting. We provide a guaranteed approach for the one–step input design problem, by selecting optimal inputs for the purpose of parameter estimation. As optimality criterion, the dimension and the outer–bounding volume of the "anticipated" consistent parameter set is considered. We furthermore derive a sufficient criterion for (one–step) parameter identifiability, i.e. when a point estimate for a parameter can be guaranteed for all possible measurements.

Keywords: parameter estimation, experimental design, identifiability, minimax, homothety, set membership

1. INTRODUCTION

Obtaining or refining a mathematical model of a dynamic process being able to reproduce available empirical data is an ubiquitous problem and a mandatory step for purposes such as prediction, analysis, or control synthesis. Very frequently however, models' parameters cannot be determined directly and have to be estimated from typically uncertain (time–series) data, while it is important to investigate the influence of this uncertainty on the parameter estimates.

A common assumption made for the uncertainty is that the data is affected by an additive random noise (Walter and Piet-Lahanier [1990], Milanese and Vicino [1991]), characterized by a known probability density function (pdf), e.g. the normal distribution (white noise). The parameter estimation problem is then considered in a statistical framework, where various techniques exist to derive an (optimal and unbiased) estimator, e.g. least squares minimization or maximum likelihood. The quality of the corresponding estimates is usually assessed by utilizing the Fisher information matrix (e.g. Ljung [1998]). In many situations, however, the probability density assumption might be questionable (Milanese and Vicino [1991]), e.g. because not enough data is available (Walter and Piet-Lahanier [1990]) or the nature of the uncertainty is dubitable.

An alternative approach, known as a set–membership or bounded error description, is to assume uncertainty to be only bounded, but otherwise unknown. Advantageously, this approach allows to derive the set of consistent parameters, rather then an isolated estimator, guaranteed to contain all possible consistent solutions. This approach was initiated by Witsenhausen [1968] and Schweppe [1968] in the domain of state estimation, and employed for parameter estimation of linear (output) systems (see e.g. Walter and Piet-Lahanier [1990], Milanese and Vicino [1991], Bai et al. [1999] and the references therein). The bounded error description has also been applied to general estimation problems of dynamic nonlinear systems, for example in regression form (Milanese and Novara [2004]), using interval analysis (see Jaulin et al. [2001] and references therein), or employing a relaxation based approach (e.g. Borchers et al. [2009]). For linear systems, as considered in this note, the consistent parameter set is polytopic, and might be very complicated. This is why many existing methods address aim to determine simple–shaped sets which are guaranteed to contain the set of consistent parameters. For this purpose, ellipsoids have been considered, e.g. in Schweppe [1968, 1973] and Fogel and Huang [1982], as well as orthotopes (Milanese and Belforte [1982]) and more general forms of simple shaped polytopes such as zonotopes (Mo and Norton [1990]).

In this contribution, we consider problems related to parameter estimation for linear, discrete time systems, in membership setting. Particularly, we first address the parameter identification and estimation problem, deriving the set of consistent parameters and its outer–bounds respectively. To this end, we describe the exact consistent parameter set recursively for a given and possibly disturbed control–state sequence, following the ideas of set–

⋆ The authors acknowledge financial support from the German Federal Ministry of Education and Research (BMBF) within the FORSYS MaCS 0313922.

dynamics employed in Artstein and Raković [2008, 2011] for control synthesis, analysis, and set invariance purposes. Then, we review classic orthotopic outer–bounding and provide a complementary approach using homothety (Raković and Fiacchini [2008]) to outer–bound the consistent parameter set by general compact and convex sets of prescribed complexity. Second, we consider the input design problem in set–membership setting; we propose an approach which allows for the selection of optimal inputs that are guaranteed to lead to a minimal volume (and least dimensional) consistent parameter set (one–step), under worst case measurement consideration. We finally relate the results to parameter identifiability question, and state a sufficient criterion when one or more model parameters can be identified exactly (in one–step), i.e. when point estimates *are* obtainable. The proposed methods are illustrated by two examples.

Paper Structure Section 2 presents necessary preliminaries. In Section 3, we present the description of the exact consistent parameter sets, and discuss outer–bounding approaches using orthotopic and homothetic sets. In Section 4, we apply and extend the obtained results to one–step input design problem and parameter identifiability in a particular case. Section 5 provides a summary and discussion.

Basic Nomenclature The sets of non–negative, positive integers and non–negative real numbers are denoted, respectively, by $\mathbb{N}, \mathbb{N}_+, \mathbb{R}_+$. We furthermore denote the integer sequence $\mathbb{N}_{[a:b]} := \{a, a+1, \dots, b\}$ with $a \in \mathbb{N}, b \in \mathbb{N}, a < b$. For a set $X \subset \mathbb{R}^n$ and a vector $y \in \mathbb{R}^n$, the Minkovski addition is defined by $y \oplus X := \{y+x \; : \; x \in X\}$. All sets considered in the remainder are compact and convex sets (unless otherwise stated). The collection of non–empty compact sets in $\mathbb{R}^n$ is denoted by $Com(\mathbb{R}^n)$. Proofs for some of the propositions are given in the appendix.

2. PRELIMINARIES AND PROBLEMS STATEMENT

In this paper, we consider linear systems of the form:

$$x_{k+1} = A(\lambda)x_k + B(\lambda)u_k + w_k, \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$ and $w_k \in \mathbb{R}^{n_x}$ are the current state, control and the unknown disturbance respectively, x_{k+1} is the successor state and $\lambda \in \mathbb{R}^{n_\lambda}$ the system parameters. The system structure is known, the matrices $A(\lambda)$, $B(\lambda)$ are given by:

$$A(\lambda) = \sum_{i=1}^{n_\lambda} A_i \lambda_i, \; B(\lambda) = \sum_{i=1}^{n_\lambda} B_i \lambda_i, \tag{2}$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{n_\lambda})$, and for all $i \in \{1, 2, \dots, n_\lambda\}$, the matrix pairs (A_i, B_i) are known and are of compatible dimension (i.e. $(A_i, B_i) \in \mathbb{R}^{n_x \times n_x} \times \mathbb{R}^{n_x \times n_u}$).

The prime concern of this note is to obtain guaranteed and as good as possible estimates of system parameters, which are unknown at the beginning of the process. The intricate case when no data is given, but an experiment is to be designed, is treated later. For parameter estimation we are given state and control measurements/sequences, possibly affected by a bounded disturbance.

We denote the bounding sets of the parameters and disturbance by Λ and W, and assume for simplicity that both sets are polytopic (compact and convex) sets in $\mathbb{R}^{n_\lambda}$ and $\mathbb{R}^{n_x}$ respectively,

$$\Lambda := \{\lambda \in \mathbb{R}^{n_\lambda} : \; M_0 \lambda \leq l_0, \; M_0^e \lambda = l_0^e\}, \tag{3}$$

$$W := \{w \in \mathbb{R}^{n_x} : \; M_w w \leq l_w\}, \tag{4}$$

with known matrix–vector pairs $(M_0, l_0) \in \mathbb{R}^{r_i \times n_\lambda} \times \mathbb{R}^{r_i}$, $(M_0^e, l_0^e) \in \mathbb{R}^{r_e \times n_\lambda} \times \mathbb{R}^{r_e}$ and $(M_w, l_w) \in \mathbb{R}^{r_w \times n_\lambda} \times \mathbb{R}^{r_w}$.

Interpretation 1. (Parameter Estimation). At the beginning of the process, the parameters λ are not known apart from being bounded, though do not change throughout the process. In contrast, the disturbances $w_k \in W$ can take different values throughout the process, known only to be bounded, and are hence not considered for estimation.

For simplicity, we consider in the remainder all states to be measured; the more general case can be found in Borchers et al. [2009], Rumschinski et al. [2010]. The available data consists of (disturbed) state and control sequences for consecutive $k \in \{1, 2, \dots, N\}$ time steps, i.e. $\{x_k\}_{k=0}^N$ and $\{u_k\}_{k=0}^{N-1}$, where the state sequence is affected by unknown disturbances $w_k \in W$. The considered parameter estimation problem takes the following form:

Problem 2. (Parameter Identification). Estimate the set $\Theta_N \subseteq \Lambda$ of parameters that is *consistent* with the given data $\{x_k\}_{k=0}^N$, $\{u_k\}_{k=0}^{N-1}$, W, i.e. estimate the *consistent parameter set*

$$\begin{aligned} \Theta_N := \{\lambda \in \Lambda : \; & \forall k \in \mathbb{N}_{[0:N-1]}, \\ & x_{k+1} = A(\lambda)x_k + B(\lambda)u_k + w_k, \\ & w_k \in W\}. \end{aligned} \tag{5}$$

Since the consistent parameter set might take complicated forms with increased number of measurements, we are interested in outer–bounding Θ_N by general polytopic shapes. Having in mind the trade–off between efficiency and precision, simple or more complicated basic shapes can be utilized. To this end, we consider families of homothetic sets (Raković and Fiacchini [2008]), defined as follows:

Definition 3. Sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ are called (positively) homothetic if $X = z \oplus \alpha Y$ for some $z \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}_+$.

For ease of notation, we denote for any state/control pair $(x_k, u_k) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ and for any $i \in \{1, 2, \dots, n_\lambda\}$,

$$\begin{aligned} & y_i(x_k, u_k) = A_i x_k + B_i u_k, \\ & Y(x_k, u_k) := (y_1(x_k, u_k) \quad y_2(x_k, u_k) \quad \dots \quad y_{n_\lambda}(x_k, u_k)), \end{aligned} \tag{6}$$

where $y_i(x_k, u_k) \in \mathbb{R}^{n_x}$ and $Y(x_k, u_k) \in \mathbb{R}^{n_x \times n_\lambda}$. Notice that, under the construction above, for any given (x_k, u_k), $Y(x_k, u_k)\lambda = A(\lambda)x_k + B(\lambda)u_k$.

With the preparations above, we can now turn our attention to the parameter identification and outer–bounding problem.

3. PARAMETER ESTIMATION

In this section, we consider the exact identification, and then outer–bounding of the consistent parameter sets, often required for a practicable analysis or synthesis problems.

3.1 Exact Characterization

Recall that the model parameter λ are known only to the extend that $\lambda \in \Lambda$ and that they do not change over time (i.e. the values of λ are equal to its values at the beginning of the process). However, the disturbances w_k are not known and it can take, at any point in time, any arbitrary value in the set W. Following the set–dynamics ideas presented in Artstein and Raković [2008, 2011], we have:

Proposition 4. The dynamic of the consistent parameter set (5) is described by

$$\Theta_{k+1} = F(\Theta_k, x_{k+1}, x_k, u_k), \tag{7}$$

where the map $F(\cdot,\cdot,\cdot,\cdot) : Com(\mathbb{R}^{n_\lambda}) \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_\lambda}$ is given by:

$$F(\Theta_k, x_{k+1}, x_k, u_k) = \{\lambda \in \Theta_k : x_{k+1} - Y(x_k, u_k)\lambda \in W\}. \tag{8}$$

Hence, parameter identification reduces to the determination of the sequence $\{\Theta_k\}_{k=1}^N$ of consistent parameter sets, for the given initial parameter set $\Theta_0 = \Lambda$, and the state $\{x_k\}_{k=0}^N$ and control $\{u_k\}_{k=0}^{N-1}$ sequences.

In the considered linear–polytopic setting, the computation of the sequence $\{\Theta_k\}_{k=1}^N$ simplifies as stated by:

Proposition 5. The consistent parameter sets Θ_k, $k \in \{1, 2, \ldots, N\}$ is given by:

$$\Theta_k = \{\lambda \in \Lambda : \ M_k\lambda \le l_k\}, \tag{9}$$

where Λ and (M_0, l_0) as in (3), and for all $j \in \{1, 2, \ldots, k\}$:

$$M_j = \begin{pmatrix} M_{j-1} \\ -M_w Y(x_{j-1}, u_{j-1}) \end{pmatrix},\ l_j = \begin{pmatrix} l_{j-1} \\ l_w - M_w x_j \end{pmatrix}, \tag{10}$$

Importantly, in the parameter estimation case, usually only very few inequalities of (9) contribute to the boundary of the consistent parameter set. Redundant constraints can be detected and dropped, e.g. following Mattheiss [1973], to obtain a minimal representation of the consistent parameter set. This improves computational efficiency when computation of outer–bounds is performed.

3.2 Outer–bounding

For the considered system class, the consistent parameter sets Θ_k, $k \in \{1, \ldots, N\}$ (9) are convex polytopes, see Prop. 5, which may become fairly complex if N is large. This is why many existing methods aim at determining a simple–shaped set S containing the set Θ_k. For this purpose, ellipsoids have been considered, e.g. Schweppe [1968, 1973], Fogel and Huang [1982], as well as orthotopes (Milanese and Belforte [1982]), and more general forms of simple shaped polytopes (Mo and Norton [1990]). We next review orthotopic outer–bounding for the present case, and afterward provide a novel outer–bounding approach based on homothety.

Orthotopic Outer–Bounding In practice, one is often interested in the uncertainty interval associated with each parameter λ_i, i.e. to bound the consistent parameter sets Θ_k by the orthotope aligned with the coordinate axes. The lower and upper bound which define the (compact) uncertainty interval of the i–th parameter, are obviously given by the values of the minimum and maximum criterion as follows:

$$\begin{aligned} \mathcal{O}_i(\Theta_k) &= \{[\underline{\lambda}_i, \overline{\lambda}_i],\ \ i \in \mathbb{N}_{[1:n_\lambda]}\}, \qquad (11)\\ \text{with} \quad & \underline{\lambda_i} = \min_{\lambda}\{\lambda_i\},\ \overline{\lambda_i} = \max_{\lambda}\{\lambda_i\} \\ & s.t.\ \lambda \in \Theta_k. \end{aligned}$$

The length of the (inner and outer) bounding interval of a parameter $\lambda_i \in \Theta_k$ is denoted by

$$\ell_i^k = \overline{\lambda}_i - \underline{\lambda}_i. \tag{12}$$

We define the bounding orthotope as the Cartesian product of all n_λ bounding intervals, i.e.

$$\mathcal{O}(\Theta_k) := \prod_{i=1}^{n_\lambda} \mathcal{O}_i(\Theta_k).$$

By definition, $\mathcal{O}(\Theta_k)$ is Lebesgue measurable (see e.g. Schneider [1993]), and its volume takes the form:

$$Vol\,(\mathcal{O}(\Theta_k)) := \prod_{i=1}^{n_\lambda} \ell_i^k, \tag{13}$$

where $Vol(\cdot) :\ Com(\mathbb{R}^n) \to \mathbb{R}_+$ is the volume map.

In (11), the computation of the collection of uncertainty intervals however requires the solution of $2n_\lambda$ linear programs (e.g. using the simplex method (see e.g. Boyd and Vandenberghe [2004]). Alternatively, the bound collection can also be obtained via a single (though larger) concave program, required later on for experimental design, as follows:

Proposition 6. The collection of bounds of Θ_k and respective volume are obtained by:

$$\lambda^* = \arg\max_{\underline{\lambda}, \overline{\lambda}} \{ \prod_{i=1}^{i=n_\lambda} (\overline{\lambda}_i^{(i)} - \underline{\lambda}_i^{(i)})\} \tag{14}$$
$$\text{s.t.} \quad \forall i \in \mathbb{N}_{[1:n_\lambda]},\ \overline{\lambda}_i^{(i)} \ge \underline{\lambda}_i^{(i)},\ \overline{\lambda}^{(i)} \in \Theta_k,\ \underline{\lambda}^{(i)} \in \Theta_k,$$

where $\lambda^* = (\underline{\lambda}^*, \overline{\lambda}^*)$ is the collection of bounds.

The volume $Vol\,(\mathcal{O}(\Theta_k))$ is simply obtained by replacing "argument" with "max" in (14). Proof immediately follows by construction.

Remark 7. Notice that $2n_\lambda$ (independent) parametrization's are introduced, denoted by $\underline{\lambda}^{(i)}$ and $\overline{\lambda}^{(i)}$ for $i \in \{1, 2, \ldots, n_\lambda\}$. Note that problem (21) can be formulated as determinant maximization problem, which is a concave problem (see e.g. Boyd and Vandenberghe [2004]).

Remark 8. Whenever $Vol\,(\mathcal{O}(\Theta_k)) = \emptyset$, $\Theta_k = \emptyset$, thus providing fact that the model (1) is invalid (inconsistent with the measurements).

Homothety A complementing alternative to fixed–shape bounding approaches is to employ homothety (Raković and Fiacchini [2008]), which provides the flexibility to consider general compact and convex shapes for outer–bounding of the sets Θ_k.

To this end, consider the family of homothetic sets:

$$\mathcal{S}(S) = \{s \oplus \alpha S, s \in \mathbb{R}^{n_\lambda}, \alpha \in \mathbb{R}_+\}. \tag{15}$$

Here $s \in \mathbb{R}^{n_\lambda}$ is an orientation vector, and $\alpha \in \mathbb{R}_+$ a scalar representing the width of the set S. The set $S \subseteq \mathbb{R}^{n_\lambda}$ is designed off–line, and can in principle be an arbitrary non–empty compact, convex set. The choice of the basic shape S however might depend on a particular application or on some quality criterion; for example, computational efficiency obliges simple basic shapes, e.g. orthotopes, whereas accuracy, i.e. advantageous relation of inner and outer approximation, typically requires more complex basic shapes.

Here, we focus on an outer–bounding map $\mathcal{O}_{\mathcal{S}}(\cdot) : Com(\mathbb{R}^{n_\lambda}) \to \mathcal{S}$ given by:

$$\mathcal{O}_{\mathcal{S}}(X) := \arg\inf_{S}\{h(X, S)\ :\ S \in \mathcal{S} \text{ and } X \subseteq S\},$$

where $h(\cdot,\cdot) : Com(\mathbb{R}^n) \times \mathcal{S} \to \mathbb{R}$ is a selection criterion for the homothetic outer bound, e.g. the scaling factor. Then, the homothetic outer–bound of the consistent parameter set Θ_k is given by $\mathcal{O}_{\mathcal{S}}(\Theta_k)$.

Exemplary, we provide the homothetic outer–bounding for the general case when the basic shape is a irreducible polytopic set $S = \{s : C_s s \leq d_s\}$ ($C_s \in \mathbb{R}^{n_s \times n_\lambda}$, $d_s \in \mathbb{R}^{n_s}$), for Θ_k (9):

$$(s_k, \alpha_k) = \arg\min_{s,\alpha}\{\alpha^2\} \tag{16}$$

$$\text{s.t.} \begin{pmatrix} C_s & d_s \\ -C_s & -d_s \end{pmatrix} \cdot \begin{pmatrix} s \\ \alpha \end{pmatrix} \leq \begin{pmatrix} C_s s_0 + d_s\alpha_0 \\ -\overline{c} \end{pmatrix},$$

where (s_0, α_0) defines the initial basic shape, and with $\overline{c} = (\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_{n_s})^T$ given by

$$\overline{c}_j = \max_{\lambda}\{C_s(j)\lambda\}$$
$$\text{s.t. } \lambda \in \Theta_k,$$

with $C_s(j)$ denoting the $j-$th row of C_s.

Remark 9. Note that by construction it holds that $\Theta_{k+1} \subseteq \Theta_k$, hence $\mathcal{O}_{\mathcal{S}}(\Theta_{k+1}) \subseteq \mathcal{O}_{\mathcal{S}}(\Theta_k)$, and therefore $\alpha_{k+1} \leq \alpha_k$, i.e. the scaling factor sequence $\{\alpha_k\}_{k=0}^N$ is monotonically non–increasing.

Illustrative Example 1

As example we consider the following uncertain linear system

$$x_{k+1} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} x_k + \begin{pmatrix} \lambda_5 \\ \lambda_6 \end{pmatrix} u_k + w_k \tag{17}$$

with $n_x = 2$, $n_u = 1$, and $n_w = 2$. The disturbances $w_k = (w_{1,k}, w_{2,k})^T$ are bounded, $0 \leq w_{1,k} \leq 0.2$, $0 \leq w_{2,k} \leq 0.2$, and the six parameters are unknown to the extend

$$\Lambda = \Theta_0 = \{\lambda \in \mathbb{R}^6 : \forall i \in \mathbb{N}_{[1:6]}, 0 \leq \lambda_i \leq 1\}.$$

We generate artificial measurements ($N = 30$) using the reference parametrization $\lambda^* = (0.1, 0.2, 0.1, 0.3, 0.2, 0.1)^T$. We consider two experiments with same initials $x_0 = (0, 0)^T$, same input sequence $u_0 = 1$, $\{u_k \sim \{0, 1\}\}_1^{29}$. Two different realizations are obtained by considering two independent random disturbance sets $\{w_{i,k}^{(i)} \sim [0, 0.2]\}_1^{29}$, $i = \{1, 2\}$, by which two sequences $\{x_k^{(i)}\}_{k=0}^{30}$, $i = \{1, 2\}$ are computed.

For this two measurement sequences, we estimate the dynamics of bounding intervals for the six parameters according to Prop. 6. The results are depicted in Fig. 1. The example demonstrates that although parameters intervals can be narrowed, the estimates quality strongly depends on the actual disturbances.

To obtain homothetic outer–bound $\mathcal{O}_{\mathcal{S}}(\Theta_k)$, we choose as basic shape a simple cube, e.g.

$$\mathcal{S} = \{s \in \mathbb{R}^6 : \forall i \in \mathbb{N}_{[1:6]}, 0 \leq s_i \leq 1\}, s_0 = 0 \in \mathbb{R}^6, \alpha_0 = 1.$$

Fig. 2 shows respectively the evolution of the homothetic scaling factor α_k, estimated according (16), for both data sets. For the considered basic shape, α_k can be interpreted as the maximum length of the uncertainty intervals, i.e. $\alpha_k = \max_{i \in \mathbb{N}_{[1:6]}} \ell_i^k$, compare Fig. 1.

4. ONE–STEP EXPERIMENTAL DESIGN AND IDENTIFIABILITY

As a second task in this paper, we consider an experimental design problem related to parameter estimation in the set–membership setting. Particularly, we aim to design inputs which are guaranteed to lead to a minimal

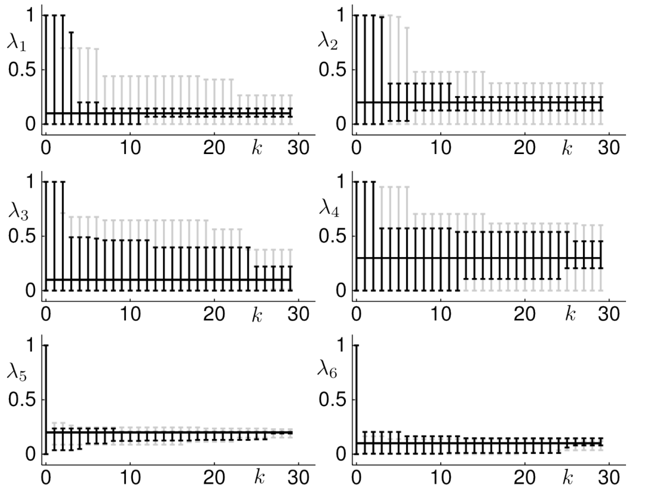

Fig. 1. Orthotopic outer bounding. Evolution of the bounding intervals $\mathcal{O}_i(\Theta_k)$ for two realizations of the same experiment, shown in different colors. Reference values are indicated by the black lines.

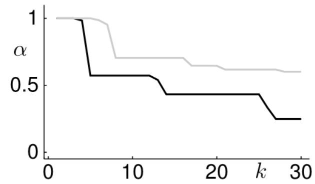

Fig. 2. Homothetic outer bounding. Evolution of the scaling factor α for the unit cube as basic shape.

volume consistent parameter set, ideally to a point estimate. Obviously, this problem is much more challenging then parameter estimation, since, apart that the unknown system parameters are bounded ($\lambda \in \Lambda$), little further information is available. The actual measurements are not known and can take any feasible value. However, we can exploit the information that the measurement will be a singleton ($z = x_{k+1} \in \mathbb{R}^{n_x}$).

While in general it is of course desired to design experiments for several future time steps, i.e. to design an optimal input trajectory/sequence, we restrict ourselves for simplicity to a single future time instance in the remainder of this note. This is basically because the single step approach can be handled using geometric programming (see e.g. Boyd and Vandenberghe [2004]) as shown later, whereas the multi–step case is more intricate due to bilinear terms, which requires solving polynomial programs (e.g. via a relaxation–based approach as in Borchers et al. [2009]), which is thereby out of the scope of this note and subject of further research.

We treat the disturbance case later, and consider for now the disturbance–free system as in (1) with fixed initial state $x_0 \in \mathbb{R}^{n_x}$ and unknown parameters $\lambda \in \Lambda$. Controls of the domain $U = \{u : u \in \mathbb{R}^{n_u}\}$ can be applied.

Problem 10. (One–step Input Design). Design an input $u^* \in U$ which leads to a minimal consistent parameter set for the worst possible measurement $z^* \in \mathbb{R}^{n_x}$, i.e. find

$$u^* = \arg\min_{u} \max_{z} \{Vol\,(\Theta_1)\}, \tag{18}$$

where $Vol(\cdot) :\ Com(\mathbb{R}^n) \to \mathbb{R}_+$, and $\Theta_1 = F(\Lambda, z, x_0, u)$,

$$\Theta_1 = \{\lambda \in \Lambda : \quad z = A(\lambda)x_0 + B(\lambda)u, \quad u \in U,\ z \in \mathbb{R}^{n_x}\}. \tag{19}$$

Notice that, in contrast to (one–step) parameter estimation, z is not known and can take any admissible value in $\mathbb{R}^{n_x}$ (for non–admissible values we have $Vol\,(\Theta_1) = \emptyset$).

Problem (18)–(19) is in general hard to solve, and to obtain the desired guaranteed results we propose the following two relaxations. First, determining the exact volume of polytopic sets is very difficult for the general case $n_\lambda \geq 3$. Therefore, we consider, for the general case, the volume of the bounding orthotope $Vol\,(\mathcal{O}(\Theta_1))$ (13) instead, although guaranties can be still provided. Second, we consider the case when the control set has a discrete domain, e.g. $U_d = \{u_j \in \mathbb{R}^{n_u},\ j \in \{1, 2, \ldots, n_d\}\}$.

With this simplifications, input design problem (18)–(19) consists now in *selecting* an input $u^* \in U_d$ which minimizes the volume of $\mathcal{O}(\Theta_1)$ for the worst possible measurement $z^* \in \mathbb{R}^{n_x}$, i.e.

$$u^* = \arg\min_{u \in U_d} \{Vol\,(\mathcal{O}(\Theta_1))^*\}, \tag{20}$$

with $Vol(\cdot) : \ Com(\mathbb{R}^{n_\lambda}) \to \mathbb{R}_+$ as in (13), and $\Theta_1 = F(\Lambda, z, x_0, u_j)$

$$\begin{aligned} Vol\,(\mathcal{O}(\Theta_1))^* \ =& \max_{z,\underline{\lambda},\overline{\lambda}} \{ \prod_{i=1}^{i=n_\lambda} (\overline{\lambda}_i^{(i)} - \underline{\lambda}_i^{(i)})\} \\ \text{s.t.}\ \ & \forall i \in \mathbb{N}_{[1:n_\lambda]}, \\ & \overline{\lambda}_i^{(i)} \geq \underline{\lambda}_i^{(i)},\ \overline{\lambda}^{(i)} \in \Lambda, \underline{\lambda}^{(i)} \in \Lambda, \\ & z = A(\overline{\lambda}^{(i)})x_0 + B(\overline{\lambda}^{(i)})u_j, \\ & z = A(\underline{\lambda}^{(i)})x_0 + B(\underline{\lambda}^{(i)})u_j, \\ & z \in \mathbb{R}^{n_x}. \end{aligned} \tag{21}$$

Analogously to (14), $2n_\lambda$ independent parametrization's are introduced, and problem (21) is log–max concave.

The proposed selection approach (20) hence requires solving n_d programs (21), summarized by:

Proposition 11. (One–step Input Selection). The input u^* (20) minimizes the volume of the outer–bounded consistent parameter set $\mathcal{O}(\Theta_1)$ (21), for all possible instances of z and every $u \in U_d$.

Remark 12. Note that for the trivial case $n_\lambda = 1$, $Vol(\mathcal{O}(\lambda))^* = Vol(\lambda)^*$, i.e. the input design problem is solved *exactly.* Also for the case $n_\lambda = 2$, where the consistent parameter set is an area whose measure can be *explicitly* described using vertex enumeration (e.g. following Avis and Fukuda [1992]), outer–bounding is not required.

Remark 13. The input selection approach directly extends to the initial condition selection problem, by treating initial conditions as inputs (see Ex. 2).

Furthermore, an important conclusion can be drawn from the case $Vol\,(\mathcal{O}(\Theta_1))^* = 0$ by (21). Then, by construction, at least one bounding interval is a singleton set, and hence a point estimate can be guaranteed for at least one parameter for all possible measurements z, i.e. one or more parameters are *identifiable.* By construction, those parameters are distinguished as follows:

Proposition 14. (One–step Identifiability). Given an disturbance free system (1), with unknown parameters $\lambda \in \Lambda$, known initial condition $x_0 \in \mathbb{R}^{n_x}$, and an input $u_j \in U_d$. If $\ell_i^1 = 0$, with

$$\begin{aligned} \ell_i^1 \ =& \max_{z,\underline{\lambda},\overline{\lambda}} \{(\overline{\lambda}_i - \underline{\lambda}_i)\} \\ \text{s.t.}\ & \overline{\lambda}_i \geq \underline{\lambda}_i,\ \overline{\lambda} \in \Lambda, \underline{\lambda} \in \Lambda, \\ & z = A(\overline{\lambda})x_0 + B(\overline{\lambda})u_j, \\ & z = A(\underline{\lambda})x_0 + B(\underline{\lambda})u_j, \\ & z \in \mathbb{R}^{n_x}, \end{aligned} \tag{22}$$

then λ_i is *identifiable* in one step by input u_j.

As a consequence, the input design problem necessitates a prior optimality criterion, e.g. the Hausdorff dimension (see e.g. Mattila [1999]) of $\mathcal{O}(\Theta_1)$, preferring inputs leading to point estimates, refer Ex. 2. To this end, the objective of (21) can be tailored to

$$Vol\,(\mathcal{O}(\Theta_1))^* = \max_{z,\underline{\lambda},\overline{\lambda}} \{ \prod_{i=1, i\neq k}^{i=n_\lambda} (\overline{\lambda}_i^{(i)} - \underline{\lambda}_i^{(i)})\}$$

to remove identifiable parameter(s) λ_k, to proceed with the optimal input selection problem (20), where now $Vol(\cdot) : \mathbb{R}^n \to \mathbb{R}_+$ with $n \leq n_\lambda$.

Extension to the Robust Case The experimental design problem (20)–(21) is based on the assumption that systems states are not perturbed. The extension to the robust case is found by

$$u^* = \arg\min_{u \in U_d} \{Vol\,(\mathcal{O}(\Theta_1))^*\}, \tag{23}$$

with $\Theta_1 = F(\Lambda, z, x_0, u_j)$, and

$$\begin{aligned} Vol\,(\mathcal{O}(\Theta_1))^* \ =& \max_{z,\underline{\lambda},\overline{\lambda}} \{ \prod_{i=1}^{i=n_\lambda} (\overline{\lambda}_i^{(i)} - \underline{\lambda}_i^{(i)})\} \\ \text{s.t.}\ \ & \forall i \in \mathbb{N}_{[1:n_\lambda]}, \\ & \overline{\lambda}_i^{(i)} \geq \underline{\lambda}_i^{(i)},\ \overline{\lambda}^{(i)} \in \Lambda, \underline{\lambda}^{(i)} \in \Lambda, \\ & -M_w Y(x_0, u_j)\underline{\lambda}^{(i)} \leq l_w - M_w z, \\ & -M_w Y(x_0, u_j)\overline{\lambda}^{(i)} \leq l_w - M_w z, \\ & z \in \mathbb{R}^{n_x}. \end{aligned} \tag{24}$$

Note that (24) can be formulated as (concave) determinant maximization problem.

Illustrative Example 2

We consider the simple linear system

$$x_1 = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 + \lambda_2 & 0 \end{pmatrix} u,$$

where $u \in \mathbb{R}^2$ denote the input vector, $z = x_1 \in \mathbb{R}^2$ the unknown (future) state, and

$$\Lambda = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : 0 \leq \lambda_1 \leq 1, 0 \leq \lambda_2 \leq 1\}$$

defining the initial parameter set. We aim designing inputs leading to a volume–minimal parameter set. For this, we consider four inputs as possible choices, see Tab. 1. Using Prop. 11 and Prop. 14, we obtain the following results:

For $u = (0, 0)^T$ and $u = (1, 1)^T$, the parameter bounds cannot be improved.

For $u = (0, 1)^T$, we have $\ell_2^1 = 0$, hence by Prop. 14 λ_2 can be identified exactly; the parameter λ_1 however can not be improved (as $\ell_1^1 = 1$).

Finally, for $u = (1, 0)^T$ both parameters can be identified exactly as $\ell_1^1 = \ell_2^1 = 0$. Hence $u = (1, 0)^T$ is selected as optimal control input.

Table 1. Anticipated volume $Vol(\mathcal{O}(\Theta_1))$ and bounding intervals ℓ_1^1, ℓ_2^1 for the considered experiments.

experiment			volume	bounding intervals	
#	u_1	u_2	$Vol(\mathcal{O}(\Theta_1))$	ℓ_1^1	ℓ_2^1
1	0	0	1	1	1
2	0	1	0	1	0
3	1	0	0	0	0
4	1	1	1	1	1

5. SUMMARY AND DISCUSSION

In this note, we addressed the set–membership approach for parameter estimation and input design considering linear systems given in state space notion that are subject to bounded disturbances. A main advantage of the set–membership approach is that a set of consistent estimates can be derived, *guaranteed* to contain all possible solutions. For the considered system class, we used set–dynamics to provide a recursive description of the exact parameter sets Θ_k, consistent with a given state/control sequence and possibly affected by bounded disturbances. To characterize the consistent parameter sets, which may become fairly complicated if N is large, we proposed to employ simple–shaped sets containing Θ_k. Particularly, we suggested a novel approach using homothety and families of shape–preserving sets of prescribed complexity. This in turn allows for a trade–off between efficiency and precision, using simple or more complicated basic shapes. The homothetic approach thus complements and generalizes the idea of using fixed shapes such as orthotopes or ellipsoids for outer–bounding purposes.

We furthermore considered input design problem in set–membership setting. We provided a guaranteed approach for the one–step input design problem (20)–(21) by selecting optimal inputs for the purpose of parameter estimation. As optimality criterion, the dimension and the outer–bounding volume of the "anticipated" consistent parameter set is considered. We furthermore derived a sufficient criterion for (one–step) parameter identifiability, i.e. when a point estimate for a parameter can be guaranteed for all possible measurements z. Although the results presented are limited to one step, they represent a sensible step toward a more comprehensive and general framework for design problems in set–membership setting. Future research will address generalization to multi–step and general input–output systems, e.g. using a relaxation based approach Borchers et al. [2009].

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APPENDIX

Proof of Prop.4:

Proof. Let $\Theta_k, x_{k+1}, x_k, u_k$ be given. By (7)–(8), we have $\Theta_{k+1} = F(\Theta_k, x_{k+1}, x_k, u_k)$ with

$$\begin{aligned}F(\Theta_k, x_{k+1}, x_k, u_k) &= \{\lambda \in \Theta : x_{k+1} - Y(x_k, u_k)\lambda \in W\}\\ &= \{\lambda \in \Theta_k : x_{k+1} - Y(x_k, u_k)\lambda = w_k,\ w_k \in W\}\\ &= \{\lambda \in \Theta_k : x_{k+1} = A(\lambda)x_k + B(\lambda)u_k + w_k,\ w_k \in W\}.\end{aligned}$$

Since $\Theta_0 := \Lambda$, it follows that $\Theta_{k+1} = F(\Theta_k, x_{k+1}, x_k, u_k)$ generates the desired sequence $\{\Theta_k\}_{k=1}^N$ of the consistent parameter sets. $\square$

Proof of Prop.5:

Proof. Pick a $j \in \{0, 1, \ldots, j, \ldots, N-1\}$ and assume that $\Theta_j = \{\lambda \in \Lambda : M_j\lambda \le l_j\}$. Then, by Prop. 4,

$$\begin{aligned}\Theta_{j+1} &= F(\Theta_j, x_{j+1}, x_j, u_j)\\ &= \{\lambda \in \Theta_j : x_{j+1} - A(\lambda)x_j - B(\lambda)u_j \in W\}.\end{aligned}$$

Hence, from the description of W (3) and Θ_j, we have:

$$\Theta_{j+1} = \{\lambda \in \Theta_j : M_{j+1}\lambda \le l_{j+1}\}$$

with M_{j+1}, l_{j+1} as in (10). Since $\Theta_0 := \Lambda$, the claim follows by induction. $\square$