Nonlinear Model Predictive Control:
A Sampled-Data Feedback Perspective

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MEINEN ELTERN UND MEINEN BRÜDERN

Contents

Summary

This work considers theoretical and implementational aspects of sampled-data open-loop nonlinear model predictive control (NMPC) of continuous time systems. In general, in model predictive control the applied input is based on the repeated solution of an optimal control problem, which spans over a certain prediction horizon into the future. Sampled-data open-loop NMPC refers to NMPC schemes, in which the optimal control problem is only solved at discrete recalculation instants, and where the resulting optimal input signal is applied open-loop in between. Various aspects and open questions in sampled-data open-loop NMPC are considered in this work. Specifically, methods for efficient implementations of NMPC are presented, and results with respect to theoretical questions such as nominal stability, compensation of computational and measurement delays, inherent robustness, and the output-feedback problem for sampled-data open-loop NMPC are derived. Most of the derived results are not limited to NMPC. They are rather applicable to a general class of sampled-data open-loop feedback control schemes.

Deutsche Kurzfassung

Einführung

Viele praktische Regelungsaufgaben verlangen neben der Stabilisierung der Strecke die Minimierung einer Kostenfunktion unter Berücksichtigung von Beschränkungen an die Prozessgrößen. Ein Beispiel hierfür ist die Regelung eines exothermen Polymerisationsprozesses unter Beachtung einer beschränkten Kühlleistung mit dem Ziel der Minimierung der eingesetzten Energie. Ein Regelungsverfahren, das diesen Anforderungen gerecht wird, ist die prädiktive Regelung.

Die prädiktive Regelung, auch modell-prädiktive Regelung oder Regelung auf einem sich bewegendem Horizont[1] gehört zur Klasse der modell-basierten Regelungsverfahren. Im Gegensatz zu herkömmlichen Regelungsverfahren, wie zum Beispiel der PI-Regelung, wird das Eingangssignal in der prädiktiven Regelung nicht nur auf der Basis des aktuellen Zustands bestimmt. Vielmehr wird das vorhergesagte Verhalten der Strecke explizit bei der Selektion des Eingangssignals berücksichtigt. Zu diesem Zweck wird das dynamische Verhalten des Systems mit Hilfe eines Prozessmodells über einen bestimmten Zeitraum in die Zukunft, dem sogenannten Prädiktionshorizont T_p, vorhergesagt (vergleiche auch Abbildung 1). Basierend auf dieser Vorhersage wird der Stellgrößenverlauf so bestimmt,

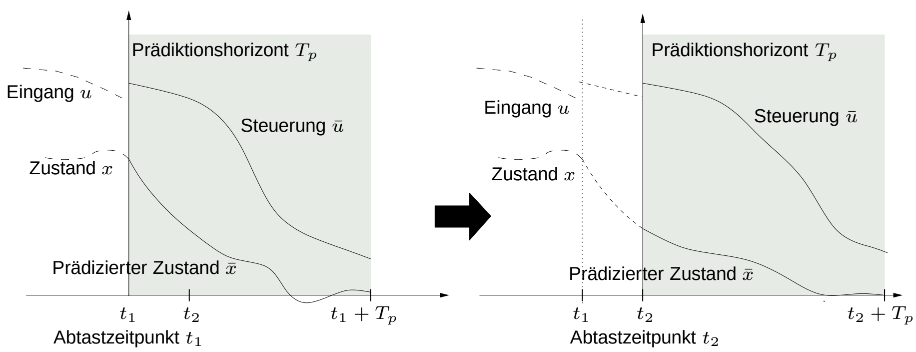


Abbildung 1: Grundprinzip der prädiktiven Regelung. Die Zeitpunkte t_i bezeichnen die Abtastzeiten und T_p den Prädiktionshorizont.

dass eine vorgegebene Kostenfunktion, in den meisten Fällen die integrierte quadratische Regelabweichung, minimiert wird. Der erste Teil des resultierenden optimalen Stellgrößenverlaufs wird als

[1] Im Englischen als *moving horizon control*, *model predictive control (MPC)* oder *receding horizon control* bezeichnet.

Steuerung auf das System aufgeschaltet und der aus Prädiktion und Minimierung der Kostenfunktion bestehende Vorgang zum nächsten Abtastzeitpunkt wiederholt.

Prinzipiell unterscheidet man zwischen linearer und nichtlinearer prädiktiver Regelung. Bei der linearen prädiktiven Regelung werden ein lineares Prozessmodell und eine quadratische Kostenfunktion verwendet, und es können lineare Beschränkungen berücksichtigt werden.

In Lauf der letzten Jahrzehnte hat sich die lineare prädiktive Regelung, vor allem in der Prozessindustrie, als eines der Standardregelungsverfahren etabliert (Qin and Badgwell, 2000; Qin and Badgwell, 2003; García et al., 1989; Morari and Lee, 1999; Froisy, 1994). So wurde bereits im Jahr 1996 von mehr als 2200 erfolgreichen industriellen Anwendungen der linearen prädiktiven Regelung berichtet (Qin and Badgwell, 1996). Schätzungen aus dem Jahr 2002 (Qin and Badgwell, 2003) gehen von mehr als 4500 industriell eingesetzten linearen prädiktiven Reglern aus. Der Einsatzbereich erstreckt sich von der Chemieindustrie über die Lebensmittelindustrie bis hin zur Luft- und Raumfahrt und der Automobilbranche. Der industrielle Erfolg der linearen prädiktiven Regelung ist auch daran ersichtlich, dass in Prozessleitsystemen der neusten Generation oft standardmäßig einfache lineare prädiktive Regelungsverfahren implementiert sind (Qin and Badgwell, 2003). Die meisten theoretischen als auch praktischen Fragestellungen auf dem Gebiet der linearen prädiktiven Regelung können als sehr gut verstanden angesehen werden (Lee and Cooley, 1996; Morari and Lee, 1999; Mayne et al., 2000).

Ständig steigende ökologische, ökonomische und sicherheitstechnische Anforderungen er-zwingen heutzutage jedoch oft, die betrachteten Prozesse über einen großen Arbeitsbereich zu betreiben. Für die Regelung solcher Prozesse spielt die Berücksichtigung auftretender Nichtlinearitäten oft eine wichtige Rolle, da für einen großen Arbeitsbereich ein lineares Modell die Realität häufig nur unzureichend wiedergibt. Aus diesem Grund ist in den letzten Jahren ein stetig wachsendes Interesse an praktisch einsetzbaren, theoretisch fundierten nichtlinearen prädiktiven Regelungsverfahren zu beobachten. Gleichzeitig wurden erhebliche Fortschritte auf dem Gebiet der nichtlinearen prädiktiven Regelung erzielt (Mayne et al., 2000; Allgöwer et al., 1999; De Nicolao et al., 2000; Qin and Badgwell, 2003; Chen and Allgöwer, 1998a; Rawlings, 2000; Allgöwer et al., 2004; Findeisen and Allgöwer, 2001; Findeisen et al., 2003d). Jedoch gibt es noch eine Reihe von Problemen, die überwunden werden müssen, bevor die nichtlineare prädiktive Regelung in der Praxis so erfolgreich und zuverlässig eingesetzt werden kann wie die lineare prädiktive Regelung. Zu den offenen Problemen gehören unter anderem:

- Die effiziente und zuverlässige Lösung des auftretenden Optimalsteuerungsproblems in Echtzeit. Sie ist eines der Schlüsselelemente für die praktische Anwendung der nichtlinearen prädiktiven Regelung.
- Die Analyse der Robustheitseigenschaften der prädiktiven Regelung, sowie die Entwicklung praktisch einsetzbarer, robust stabilisierender prädiktiver Regelungsverfahren.
- Die Entwicklung von prädiktiven Ausgangsregelungsverfahren, die die Stabilität des geschlossenen Kreises garantieren können.

Im Rahmen dieser Arbeit werden Antworten und Lösungen zu einigen dieser offenen Fragen und Probleme gegeben. Unter anderem werden effiziente Lösungsmethoden für das sich ergebende Optimalsteuerungsproblem aufgezeigt und Untersuchungen bezüglich der nominellen Stabilität, sowie der

Berücksichtigung möglicher auftretender Verzögerungen, der Robustheit des geschlossenen Kreises, und des Ausgangsregelungsproblems durchgeführt. Die erzielten Ergebnisse beschränken sich nicht auf die nichtlineare prädiktive Regelung. Vielmehr sind die meisten Ergebnisse allgemein für Regelungsverfahren gültig, die auf abgetasteten Zustandsinformationen und der Anwendung von *open-loop* Eingangssignalen beruhen.

Grundlagen

Es wird angenommen, dass der zu stabilisierende Prozess durch ein gewöhnliches, zeitinvariantes nichtlineares Differentialgleichungssystem der Form

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \tag{I}$$

beschrieben wird. Hier ist $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ der Zustandsvektor und $u(t) \in \mathcal{U} \subset \mathbb{R}^m$ der Eingangsvektor. Die Mengen $\mathcal{U}$ ist die Menge der zulässigen Eingangswerte und die Menge $\mathcal{X}$ beschreibt die erlaubten Systemzustände. Es wird angenommen, dass $\mathcal{U}$ eine kompakte und $\mathcal{X}$ eine einfach zusammenhängende Menge ist. Bezüglich des Vektorfeldes $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ wird angenommen, dass es lokal Lipschitz-stetig im Systemzustand x und stetig in u ist. Zusätzlich gelte $(0,0) \in \mathcal{X} \times \mathcal{U}$ und $f(0,0) = 0$.

Die Berechnung des Eingangssignals erfolgt nur an diskreten Abtastzeitpunkten t_i. Bezüglich der Abtastzeitpunkte wird angenommen, dass die Zeitspanne $\delta_i = t_{i+1} - t_i$ zwischen zwei Abtastzeitpunkten t_i und t_{i+1} endlich ist und nicht verschwindet.

In der prädiktiven Regelung ist das Eingangssignal, das zwischen den Abtastzeitpunkten *open-loop* auf das System aufgeschaltet wird, im Allgemeinen durch die Lösung eines Optimalsteuerungsproblems der folgenden Form gegeben:

$$\min_{\bar{u}(\cdot)} \quad J(\bar{x}(\cdot), \bar{u}(\cdot)) \tag{IIa}$$

$$\text{unter den Nebenbedingungen:} \quad \dot{\bar{x}}(\tau) = f(\bar{x}(\tau), \bar{u}(\tau)), \quad \bar{x}(t_i) = x(t_i), \tag{IIb}$$

$$\bar{u}(\tau) \in \mathcal{U}, \ \bar{x}(\tau) \in \mathcal{X} \quad \tau \in [t_i, t_i + T_p], \tag{IIc}$$

$$\bar{x}(t_i + T_p) \in \mathcal{E}. \tag{IId}$$

Hier ist J die betrachtete Kostenfunktion, die über dem Vorhersagehorizont T_p ausgewertet wird. Die Größe $\bar{x}$ stellt den vorhergesagten Zustandsverlauf des Systems (I) ausgehend vom Systemzustand $x(t_i)$ unter dem Stellgrößenverlauf $\bar{u}(\cdot)$ über das Vorhersagefenster $[t_i, t_i + T_p]$ dar. Die Unterscheidung zwischen den vorhergesagten Systemzuständen $\bar{x}$ und dem realen Systemzustand x ist notwendig, da diese sich sogar im nominellen Fall bei Verwendung eines endlichen Vorhersagehorizonts unterscheiden. Die Endbedingung (IId) erzwingt, dass der letzte vorhergesagte Systemzustand in der Endregion $\mathcal{E}$ liegt. Die Kostenfunktion J ist im Allgemeinen durch

$$J(x(\cdot), u(\cdot)) = \int_{t_i}^{t_i + T_p} F(x(\tau), u(\tau)) d\tau + E(x(t_i + T_p)) \tag{IIe}$$

gegeben. Hierbei ist F eine im Systemzustand x positiv definite Funktion, die oft auf ökologischen und ökonomischen Betrachtungen beruht. Das Endgewicht E wird zusammen mit der Endbedingung (IId) oft dazu genutzt, die Stabilität des geschlossenen Kreises zu erzielen oder die Regelgüte zu verbessern. Das auf das System aufgeschaltete Eingangssignal ist durch die folgende Beziehung definiert:

$$u(t) = \bar{u}^{\star}(t; x(t_i)). \tag{III}$$

Hier ist $u^{\star}(\cdot; x(t_i))$ das optimale Eingangssignal des Optimalsteuerungsproblems (II) für den Zustand $x(t_i)$ zum unmittelbar vorhergegangenen Abtastzeitpunkt. Das angewendete Eingangssignal basiert also auf einer wiederholten Lösung des Optimalsteuerungsproblems (II) zu den Abtastzeitpunkten t_i.

Es existieren eine Reihe nichtlinearer prädiktiver Regelungsverfahren, bei denen durch geeignete Wahl des Prädiktionshorizonts T_p, des Gewichtsterms F, des Endgewichts E und der Endregion $\mathcal{E}$ die Stabilität des nominellen geschlossenen Kreises garantiert werden kann. Details hierzu können zum Beispiel (Mayne et al., 2000; Allgöwer et al., 1999; Fontes and Magni, 2003; Chen and Allgöwer, 1998a; Findeisen et al., 2003d) entnommen werden.

Effiziente numerische Implementation

Für den praktischen Einsatz der nichtlinearen prädiktiven Regelung ist die effiziente Problemformulierung und Lösung des auftretenden Optimalsteuerungsproblems in Echtzeit von erheblicher Bedeutung. Eines der Hauptargumente gegen den praktischen Einsatz der nichtlinearen prädiktiven Regelung ist, dass das Optimalsteuerungsproblem (II) für die meisten Regelungsprobleme nicht schnell und zuverlässig genug gelöst werden kann (Qin and Badgwell, 2003). Im Rahmen dieser Arbeit wird mit Hilfe von Simulationen und experimentellen Ergebnissen für die Regelung einer Destillationskolonne zur hochreinen Trennung von Methanol und n-Propanol exemplarisch nachgewiesen, dass die Lösung des auftretenden Optimalsteuerungsproblems in Echtzeit auch mit der heute zur Verfügung stehenden Rechenleistung möglich ist. Es ist dazu notwendig, vorhandene effiziente dynamische Optimierungsverfahren an die speziellen Strukturen des Optimalsteuerungsproblems, das in der nichtlinearen prädiktiven Regelung auftritt, anzupassen. Des Weiteren sollten nichtlineare prädiktive Regelungsverfahren zum Einsatz kommen, die eine effiziente Lösung, zum Beispiel durch Vermeidung von langen Prädiktionshorizonten und strikten Endbedingungen, erlauben. Ein Beispiel für ein geeignetes prädiktives Regelungsverfahren ist die so genannte *quasi-infinite horizon* nichtlineare prädiktive Regelung (Chen and Allgöwer, 1998b). Das verwendete, speziell auf die Bedürfnisse der prädiktiven Regelung angepasste, dynamische Echtzeitoptimierungsverfahren basiert auf einem speziellen Mehrzielverfahren (Bock and Plitt, 1984; Bock, Diehl, Leineweber and Schlöder, 2000). Dieses wurde im Rahmen einer Studie über die technische Realisierbarkeit (Diehl, Findeisen, Nagy, Bock, Schlöder and Allgöwer, 2002; Diehl et al., 2001; Findeisen, Allgöwer, Diehl, Bock, Schlöder and Nagy, 2000; Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2002) der nichtlinearen prädiktiven Regelung am Institut für wissenschaftliches Rechnen der Universität Heidelberg entwickelt (Diehl, 2002; Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2002). Die erzielten Ergebnisse weisen nach, dass die für den Einsatz der prädiktiven Regelung zu Verfügung stehenden numerischen Lösungsverfahren und die heutzutage vorhandene Rechenleistung nicht mehr der limitierende Faktor für einen praktischen Einsatz der nichtlinearen prädiktiven Regelung sind.

Stabilitätsbedingungen für Abtastregler

Zur Betrachtung des Einflusses von Störungen und Modellunsicherheiten, sowie des Ausgangsregelungsproblems erweist es sich als zweckmäßig, sich nicht nur auf die nichtlineare prädiktive Lösung zu beschränken. Vielmehr ist es sinnvoll, allgemeine Abtastregelungen, die *open-loop* Eingangssignale verwenden, zu betrachten. Zu diesem Zweck werden, basierend auf Ideen aus der nichtlinearen prädiktiven Regelung, in einem ersten Schritt Stabilitätsbedingungen für Abtastregelungen, die *open-loop* Eingangssignale verwenden, hergeleitet. Abbildung 2 zeigt den hierbei betrachteten Aufbau. Ähnlich der prädiktiven Regelung wird davon ausgegangen, dass basierend auf

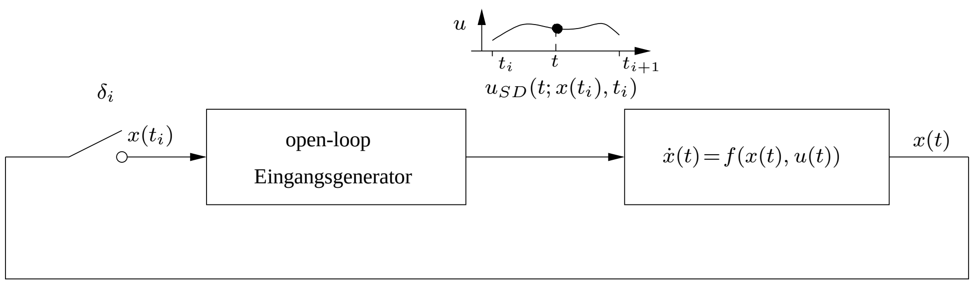


Abbildung 2: Abtastregelung unter Verwendung von *open-loop* Eingangssignalen, die durch einen Eingangsgenerator u_{SD} zu den Abtastzeiten t_i erzeugt werden.

der Zustandsinformation zum Abtastzeitpunkt t_i ein *open-loop* Eingangsgenerator ein Eingangssignal erzeugt, das bis zum nächsten Abtastzeitpunkt auf das System aufgeschaltet wird. Der geschlossene Kreis ist somit durch

$$\dot{x}(t) = f(x(t), u_{SD}(t; x(t_i), t_i)), \quad x(0) = x_0 \tag{IV}$$

gegeben. Hier bezeichnet u_{SD} das durch den *open-loop* Eingangsgenerator zum unmittelbar vor dem Zeitpunkt t liegenden Abtastzeitpunkt t_i erzeugte Eingangssignal.

Basierend auf Ideen der nichtlineare prädiktive Regelung (Fontes, 2000b; Chen and Allgöwer, 1998b; Jadbabaie et al., 2001) werden Bedingungen hergeleitet, die die Stabilität des geschlossenen Kreises im Sinne von Konvergenz zu einer vorgegebenen Zielmenge garantieren. Insbesondere erlauben die hergeleiteten Ergebnisse die Betrachtung von Systemen, die sich nur mit Hilfe von Rückführungen, die unstetig als Funktion des Systemzustandes sind, stabilisieren lassen (Brockett, 1983; Fontes, 2003; Clark, 2001; De Luca and Giuseppe, 1995; Astolfi, 1996; Ryan, 1994).

Im Gegensatz zu herkömmlichen Betrachtungen der Abtastregelung[2], in denen der Eingang zwischen den Abtastzeitpunkten konstant gehalten wird, wird in dieser Arbeit davon ausgegangen, dass das

[2] Für einen Überblick über Arbeiten auf dem Gebiet der Abtastregelung siehe zum Beispiel (Nešić and Teel, 2001; Nešić and Laila, 2002; Chen and Francis, 1995)

Ausgangssignal (nahezu) kontinuierlich implementiert werden kann. Ein Argument für diese Betrachtung ist, dass bei langen Abtastzeiten δ_i, die beispielsweise durch langsame Zustands- oder Ausgangsmessungen verursacht werden, die Fixierung des Eingangssignals auf einen konstanten Wert zu erheblichen Einbußen der Regelgüte führen kann (Nešić and Teel, 2001).

Die hergeleiteten Bedingungen werden beispielhaft dazu verwendet, Stabilitätsaussagen für ein verallgemeinertes nichtlineares prädiktives Regelungsverfahren herzuleiten. Dieses erlaubt unter anderem die Betrachtung der Stabilisierung einer Zielmenge sowie die Betrachtung unstetige Eingangssignale. Ferner wird nachgewiesen, dass aus asymptotisch stabilisierenden, lokal Lipschitz-stetigen Rückführungen durch Vorwärtssimulation des geschlossenen Kreises eine stabilisierende Abtastregelung erzeugt werden kann.

Des Weiteren wird die Problematik der in der Praxis häufig auftretenden Mess-, Rechen-, und Kommunikationsverzögerungen betrachtet. Hierfür werden einfach zu implementierende Methoden aufgezeigt, die im Fall der Abtastregelung die Berücksichtigung solcher Verzögerungen ermöglichen. Die Berücksichtigung von Verzögerungen ist insbesondere bei der nichtlinearen prädiktiven Regelung wichtig, da die Lösung des auftretenden Optimalsteuerungsproblems oftmals eine nicht zu vernachlässigende Zeit erfordert und somit zu Verzögerungen bei der Bereitstellungen des neuen Eingangssignals führt. Wird diese Rechenverzögerung nicht berücksichtigt, kann es leicht zur Instabilität des geschlossenen Kreises kommen (Findeisen and Allgöwer, 2004a). Die erzielten Ergebnisse werden anhand einer Simulationsstudie für die Regelung eines Rührkesselreaktors verifiziert.

Analyse der Robustheit von Abtastreglern

Die Analyse des Einflusses von externen Störungen und Modellfehlern ist für den praktischen Einsatz von Abtastregelungen, insbesondere für die nichtlineare prädiktive Regelung, von erheblicher Bedeutung. Die Bestimmung des *open-loop* Eingangssignals nur an den Abtastzeitpunkten hat zwar einerseits den Vorteil, dass die Zustandsinformation nur an den Abtastzeitpunkten vorliegen muss, andererseits wird die Zustandsinformation natürlich auch nur zu den Abtastzeitpunkten zurückgeführt. Im geschlossenen Kreis kann somit auf Störungen nur zu den Abtastzeitpunkten reagiert werden. Da für bestimmte Modellklassen schon beliebig kleine Fehler zu Instabilität des geschlossenen Kreises führen können (Grimm et al., 2003a; Magni et al., 2003; Findeisen et al., 2003d), ist es wichtig zu untersuchen, unter welchen Bedingungen Abtastregelungen inhärente Robustheitseigenschaften aufweisen. Im Fall der nichtlinearen prädiktiven Regelung sind solche Untersuchungen wichtig, da bisher bekannte Reglerentwürfe, die eine explizite Berücksichtigung von Störungen und Modellfehlern erlauben, praktisch nicht implementiert werden können (Fontes and Magni, 2003; Chen et al., 1997; Magni, Nijmeijer and van der Schaft, 2001).

Zur Untersuchung des Einflusses von externen Störungen und Modellfehlern geht man von einem stabilisierenden Abtastregler u_{SD} aus, der das System in einem Einzugsbereich $\mathcal{R}$ stabilisiert und eine lokal Lipschitz-stetige Wertefunktion bzw. ”Ljapunowfunktion” besitzt. Als einfachster Fall wird zunächst der Einfluss einer additiven Störung der Form

$$\dot{x}(t) = f(x(t), u_{SD}(t; x(t_i), t_i)) + p(t). \quad \text{(V)}$$

betrachtet. Hier stellt $p(t)$ den Störeinfluss dar. Für diese Störung wird nachgewiesen, dass für feste, aber beliebige kompakte Mengen Ω_γ, Ω_{c_0}, Ω_c, mit $\Omega_\gamma \subset \Omega_{c_0} \subset \Omega_c \subset \mathcal{R}$ und die durch Höhenlinien der Wertefunktion begrenzt werden (siehe Abbildung 3), immer eine Schranke $p_{\max}$ für die erlaubte Störung p existiert, so dass gilt: Wenn die Störung p für alle t_i die Bedingung

$$\left\| \int_{t_i}^{t_i+\tau} p(s) ds \right\| \leq p_{\max}\tau \quad \forall \tau \in [0, t_{i+1} - t_i], \quad \text{(VI)}$$

erfüllt, so folgt, dass für alle Anfangsbedingungen $x_0 \in \Omega_{c_0}$: 1.) die Lösung von (V) für alle Zeiten existiert, 2.) $x(t)$ die Menge Ω_c nicht verlässt, 3.) $x(t_i) \in \Omega_{c_0}$ $\forall i \geq 0$, und 4.) es eine endliche Zeit T_γ gibt, so dass $x(\tau) \in \Omega_\gamma$ $\forall \tau \geq T_\gamma$. Eine Verallgemeinerung auf Störungen, die von den Zuständen und

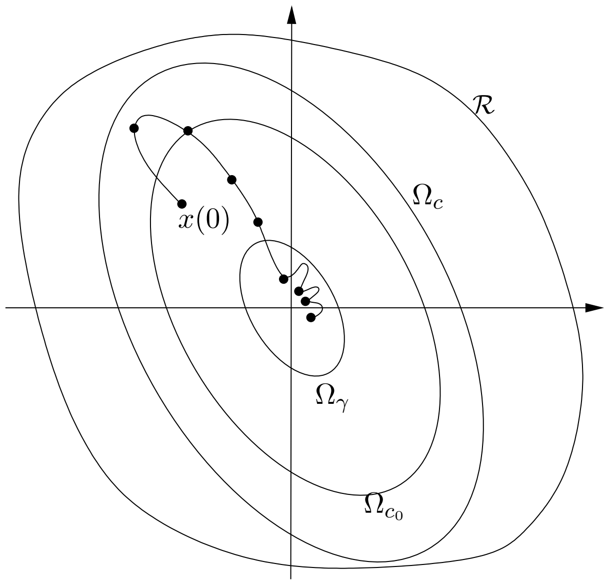


Abbildung 3: Menge der Anfangsbedingungen Ω_{c_0}, maximal zulässige Menge Ω_c, Konvergenzmenge Ω_γ und nomineller Einzugsbereich des Abtastreglers $\mathcal{R}$.

dem Eingangssignal abhängen, ist möglich, solange die Bedingung (VI) erfüllt ist. Dieses Ergebnis ist von praktischem Interesse, da es nachweist, dass hinreichend kleine Störungen im Sinne praktischer Stabilität toleriert werden können. Unter weiteren Annahmen ist es möglich, dieses Ergebnis auf Störungen auszuweiten, die direkt auf das Eingangssignal wirken, d.h. bei denen der geschlossene Kreis durch

$$\dot{x}(t) = f(x(t), u_{SD}(t; x(t_i), t_i) + v(t)) \quad \text{(VII)}$$

beschrieben wird, wobei v der auftretenden Eingangsstörterm ist. Das erzielte Ergebnis erlaubt beispielsweise die Robustheit bezüglich kleiner numerischer Fehler bei der Lösung des Optimalsteuerungsproblems zu untersuchen und die Robustheit im Fall schneller, bei der Modellbildung vernachlässigter Aktuatordynamiken sicherzustellen. In ähnlicher Weise ist es möglich, die Robustheit von Abtastregelungen bezüglich Fehlern bei der Zustandsschätzung beziehungsweise Messstörungen nachzuweisen. Dies legt die Grundlage für Ausgangsregelungsverfahren, die semi-regionale praktische Stabilität des geschlossenen Kreises erzielen.

Ausgangsregelung unter Verwendung von Abtastreglern

Für die bisherigen Betrachtungen wurde angenommen, dass die vollständige Zustandsinformation zur Verfügung steht. In der Praxis sind jedoch oft nicht alle Zustände messbar. Vielmehr stehen nur bestimmte Messungen zur Verfügung, die durch

$$y = h(x, u) \tag{VIII}$$

gegeben sind. In der Praxis wird dieses Problem meist durch Einsatz des so genannten *certainty-equivalence* Prinzips gelöst, d.h. für die Rückführung wird anstatt des realen Systemzustandes ein mit Hilfe eines geeigneten Beobachters geschätzter Systemzustand verwendet. Da es für nichtlineare Systeme, im Gegensatz zu linearen Systemen, kein allgemeingültiges Separationsprinzip gibt, kann aus der getrennten Stabilität des verwendeten Beobachters sowie des Abtastreglers nicht die Stabilität des geschlossenen Kreises gefolgert werden. Aus diesem Grund werden für den Fall einer lokal Lipschitz-stetigen Wertefunktion Stabilitätsbedingungen an den verwendeten Beobachter hergeleitet, die semi-regionale praktische Stabilität des geschlossenen Kreises garantieren. Die wesentliche Anforderung an den verwendeten Beobachter ist hierbei, dass für jeden noch so kleinen (erwünschten) maximalen Beobachterfehler und jede noch so kleine Konvergenzzeit Beobachterparameter existieren, so dass der Beobachterfehler nach der Konvergenzzeit diesen Beobachterfehler unterschreitet. Im Allgemeinen ist diese Anforderung nicht erfüllt. Jedoch existieren eine Reihe von Beobachterentwurfsverfahren, die dies garantieren. Beispiele sind klassische *high-gain* Beobachter (Tornambè, 1992), so genannte *moving horizon* Beobachter mit Kontraktionsnebenbedingung (Michalska and Mayne, 1995), Beobachter, die eine endliche Konvergenzzeit garantieren (Drakunov and Utkin, 1995; Engel and Kreisselmeier, 2002; Menold et al., 2003), sowie Beobachter, die eine lineare Fehlerdynamik aufweisen und bei der die Pole beliebig festgelegt werden können. Diese können zum Beispiel auf Normalformbetrachtungen und einer Ausgangsaufschaltung beruhen (Bestle and Zeitz, 1983; Krener and Isidori, 1983). Die erzielten Ergebnisse können, ähnlich dem Fall der nicht abgetasteten Zustandsrückführung (Teel and Praly, 1995; Atassi and Khalil, 1999), als ein spezielles Separationsprinzip für die Abtastregelung mit *open-loop* Eingangssignalen unter Verwendung von Zustandsbeobachtern betrachtet werden. Zwar sind die erzielten Ergebnisse nicht direkt zur Auslegung eines Ausgangsreglers geeignet, jedoch untermauern sie theoretisch, dass der geschlossene Kreis semi-regionale praktische Stabilität aufweisen kann, wenn der verwendete Abtastregler eine lokal Lipschitz-stetige Wertefunktion aufweist und ein entsprechender Beobachter zum Einsatz kommt.

Die hergeleiteten Ergebnisse werden mit Hilfe von Simulationsergebnissen für zwei Beispielsysteme, der Stabilisierung eines Pendels auf einem Wagen sowie der Regelung eines Bioreaktors veranschaulicht. In beiden Fällen werden klassische *high-gain* Beobachter sowie ein nichtlinearer prädiktiver Regler verwendet.

Zusammenfassung

Ausgangspunkt der vorliegenden Arbeit ist die Frage, inwieweit die nichtlineare prädiktive Regelung prinzipiell in der Praxis, d.h. unter nicht idealisierten Bedingungen, anwendbar ist. Hierzu wurde zum einen nachgewiesen, dass das in der nichtlinearen prädiktiven Regelung auftretenden Optimalsteuerungsprobleme unter Verwendung geeigneter Lösungsverfahren hinreichend schnell gelöst werden kann.

Zum anderen wurde die Frage der inhärenten Robustheit sowie des Ausgangsregelungsproblems im Rahmen einer verallgemeinerten Betrachtungsweise, nämlich der Abtastregelung unter Verwendung von *open-loop* Eingangssignalen, untersucht. Diese Betrachtungsweise erlaubt eine elegante Untersuchung entscheidender Fragen, die sich bei der praktischen Umsetzung der nichtlinearen prädiktiven Regelung ergeben. Neben der Analyse der inhärenten Robustheit wurde insbesondere ein neuer Zugang zu dem bisher nur unbefriedigend gelösten, praktisch bedeutsamen Problem der Ausgangsregelung aufgezeigt. Die meisten der vorgestellten Ergebnisse sind nicht auf die prädiktive Regelung beschränkt. Vielmehr sind sie unter gewissen Voraussetzungen allgemein auf Abtastregelungen unter Verwendung von *open-loop* Eingangssignalen übertragbar.

Chapter 1

Introduction

Typical objectives for controller design are the stability of the closed-loop while minimizing a desired cost function and satisfying constraints on the process variables. One classical approach taking these objectives directly into account is optimal feedback control. However, as is well known, it is often very hard, if not impossible, to obtain a closed solution for the optimal control problem describing the feedback. One possibility to circumvent the closed solution is the application of model predictive control (MPC), often also referred to as receding horizon control or moving horizon control. Basically, in model predictive control an optimal control problem is solved for the current system state. The first part of the resulting optimal input signal is applied open-loop to the system until the next recalculation instant, at which the optimal control problem for the new system state is resolved. Since the optimal control problem must only be solved for the current system state, the solution is much easier to obtain. An often intractable problem is replaced by a tractable one.

In general one distinguishes between linear and nonlinear model predictive control (NMPC). Linear MPC refers to MPC schemes that are based on linear models of the system and in which linear constraints on process variables and a quadratic cost function can be used. NMPC refers to MPC schemes that use nonlinear models for prediction and that allow to consider a non-quadratic cost-functional and nonlinear constraints on the process variables. By now linear MPC is widely used in industrial applications (Qin and Badgwell, 2000; Qin and Badgwell, 2003; García et al., 1989; Morari and Lee, 1999; Froisy, 1994). For example (Qin and Badgwell, 2003) report more than 4500 applications of linear MPC spanning a wide range from the production of chemicals to aerospace industries. Also many theoretical and implementation issues of linear MPC have been studied and are well understood (Lee and Cooley, 1996; Morari and Lee, 1999; Mayne et al., 2000).

Increasing product quality specifications and productivity demands, tighter environmental regulations and demanding economical considerations require the operation of processes over a wide range of operating conditions and often near the boundary of the admissible region. Under these conditions linear models are often not sufficient to describe the process dynamics adequately and nonlinear models must be used. This inadequacy of linear models, together with the desire of many companies to use already available nonlinear models for control, is one of the motivations for the increasing interest in nonlinear model predictive control.

In recent years much progress in the area of NMPC has been achieved, for details see Chapter 2 and (Mayne et al., 2000; Allgöwer et al., 1999; De Nicolao et al., 2000; Qin and Badgwell, 2003; Chen and Allgöwer, 1998a; Rawlings, 2000; Allgöwer et al., 2004; Findeisen and Allgöwer, 2001; Findeisen et al., 2003d). However, there remain a series of open questions and hurdles that must be overcome before a theoretically well founded practical application of NMPC is possible. Examples of open questions are the efficient and reliable online implementation of NMPC, the analysis of the inherent robustness properties of NMPC, the development of robust NMPC approaches, the compensation of delays, and the design of output-feedback NMPC approaches. Answers and solutions to some of these questions are provided in this thesis.

1.1 NMPC and Sampled-data Open-loop Feedback

We focus on NMPC for continuous time systems subject to *sampled state information*; i.e. we consider the stabilization of continuous time systems by repeatedly applying *open-loop* input trajectories obtained from the solution of an optimal control problem at *discrete recalculation instants* (compare Figure 1.1.) In the following we refer to this NMPC implementation as sampled-data

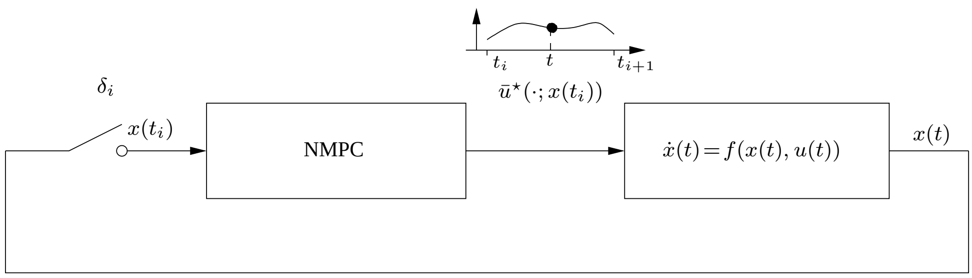


Figure 1.1: Sampled-data open-loop state-feedback using NMPC. The recalculation instants are denoted by t_i, and $\bar{u}^\star(\cdot; x(t_i))$ is the optimal open-loop input provided by the NMPC controller at the time t_i based on the state information $x(t_i)$.

open-loop NMPC. The notion of sampled-data open-loop feedback is explicitly used, since we do not consider the use of sample-and-hold elements at the input side, as classically considered in sampled-data feedback control (Chen and Francis, 1995; Aström and Wittenmark, 1997; Franklin et al., 1998; Nešić and Teel, 2001). Note that in difference to NMPC for discrete time systems (see e.g. (Mayne et al., 2000; Allgöwer et al., 1999; De Nicolao et al., 2000)) or instantaneous NMPC (Mayne et al., 2000), where the optimal input is recalculated at all times (no open-loop input signal is applied to the system), in sampled-data open-loop NMPC the behavior in between the recalculation instants must be explicitly taken into account. Sample-and-hold implementations are actually a subclass of sample-data open-loop NMPC implementations.

While instantaneous NMPC formulations, discrete time NMPC formulations, or sampled-data NMPC formulations considering a fixed input in between the recalculation times, are often appealing from a theoretical side, there are a series of practical and theoretical reasons for the application of sampled-data open-loop NMPC:

- Discrete time NMPC formulations require a discrete time process model. However, to capture the inherent nonlinearity of a process sufficiently, it is often necessary to use a first principle modeling approach, which typically leads to a set of nonlinear differential or nonlinear differential algebraic equations. Furthermore, for many processes first principle nonlinear models are already available, and companies desire to use these models directly for control. Obtaining a suitable discrete time model from continuous time models, without an implicit solution of the underlying differential equations, is often impossible.

- Fixing the input in between recalculation instants can lead to a drastic performance limitation or even instability, if the time between the recalculation instants is long. Often it is assumed that the recalculation time can be made sufficiently small to avoid such effects. However, this it not always possible, for example in the case of rarely available state and measurement information due to slow sensors, or due to extensive preprocessing. In such cases applying an open-loop input signal instead of a fixed input in between the recalculation times allows to increase the performance of the closed-loop.

- A sampled-data formulation of NMPC is practically often necessary, since the solution of the underlying optimal control problem does typically require a non-negligible amount of time, making an instantaneous implementation impossible.

- As is shown, sampled-data formulations allow a simple consideration of measurement, computational, and communication delays which are often present in practice. Not taking such delays into account can significantly decrease the performance or might even lead to instability.

To facilitate a theoretically well founded practical application of NMPC, it is important to perform a careful analysis of implementational and computational aspects of sampled-data open-loop NMPC. Even so a series of issues related to sampled-data open-loop NMPC have been considered by now, there remain many issues which have not been addressed satisfactorily or which have not been addressed at all.

The goal of this thesis is to investigate and propose solutions to some crucial open theoretical and practical aspects of sampled-data open-loop NMPC. Specifically we consider questions of:

- An efficient solution of the optimal control problem appearing in sampled-data open-loop NMPC.

- The derivation of generalized stability conditions for open-loop sampled-data feedback, including sampled-data open-loop NMPC as a special case.

- The inherent robustness properties of sampled-data open-loop feedbacks with respect to small external disturbances and model-plant mismatch, and the implications of these properties for NMPC.

- The derivation of sampled-data open-loop output-feedback schemes allowing to achieve non-local stability results.

1.2 Contribution

The area of NMPC can be considered as very fertile and has experienced a rapid development over the recent fifteen years. Nevertheless, there are a number of distinct contributions and novel viewpoints which form the core of this thesis. They contribute to the following four subgroups:

Real-time feasibility of NMPC

- A proof of concept that NMPC can be applied to realistically sized, practically relevant control problems is given. To achieve this, specially tailored numerical solution strategies together with NMPC formulations requiring a reduced computational load are used.
- The experimental verification of the derived methods by means of the control of a high-purity distillation column.

Generalized stability conditions for sampled-data open-loop state-feedback

- The derivation of generalized stability conditions for open-loop sampled-data feedback, motivated by ideas from stability proofs of NMPC, but which are not limited to sampled-data open-loop NMPC.
- The derivation of a new feedforward simulation based technique allowing to adapt any instantaneous, locally Lipschitz continuous state-feedback to the sampled-data open-loop feedback case, without loss of stability.
- The statement of a new, generalized stability theorem for sampled-data open-loop NMPC, which allows to consider the stabilization with respect to a set.
- The derivation of delay compensation techniques for sampled-data open-loop feedback retaining stability and performance of the closed-loop.

Inherent robustness properties of sampled-data open-loop state-feedback

- Analyses of the inherent robustness properties of sampled-data open-loop feedback for locally Lipschitz value/decreasing functions.
- The derivation of stability results with respect to small uncertainties and model plant mismatch (Section 5). Specific examples are the robustness with respect to small measurement errors and the robustness with respect to input disturbances or numerical errors in the solution of the optimal control problem.

Sampled-data open-loop output-feedback approaches

- The derivation of a novel output-feedback result for sampled-data open-loop feedback. Specifically stability conditions guaranteeing that the combination of a sampled-data open-loop state-feedback and a state observer achieve semi-regional practical stability are derived.

The core of this thesis is formed by the sampled-data open-loop feedback considerations presented in Chapter 4-5, and the output-feedback results presented in Chapter 6. Even so most of the derived results are clarified considering specifically NMPC, they are *not* limited to sampled-data open-loop NMPC. They rather apply to a wide class of sampled-data open-loop feedback strategies.

1.3 Thesis Outline

The thesis is structured as follows:

Chapter 2 provides an introduction and a review of existing work in the area of NMPC. The chapter is not intended to provide an overall coverage of NMPC. It is rather thought to provide the conceptual and notational basis and motivation for the considerations later on. Special emphasis is put on the differences between sampled-data open-loop feedback, the main properties, advantages and drawbacks of NMPC, implementation related issues, and system theoretical aspects of NMPC. The presentation is based on the work presented in (Findeisen et al., 2003d; Findeisen et al., 2003e; Findeisen and Allgöwer, 2001; Allgöwer et al., 2004; Allgöwer et al., 2000).

Chapter 3 summarizes results related to an efficient solution of the optimal control problem appearing in NMPC. It is shown that a real-time application of NMPC is possible if a "symbiosis" of specially tailored dynamic optimization strategies and NMPC schemes with a reduced computational load are used. After a short review of general solution methods for the optimal control problem appearing in NMPC, a specially tailored dynamic optimization strategy based on multiple shooting methods is outlined. This strategy was developed in the context of a computational feasibility study of NMPC (Nagy, Findeisen, Diehl, Allgöwer, Bock, Agachi, Schlöder and Leineweber, 2000; Findeisen, Allgöwer, Diehl, Bock, Schlöder and Nagy, 2000; Bock, Diehl, Schlöder, Allgöwer, Findeisen and Nagy, 2000; Diehl, 2002; Diehl, Findeisen, Nagy, Bock, Schlöder and Allgöwer, 2002; Diehl et al., 2001; Findeisen, Nagy, Diehl, Allgöwer, Bock and Schlöder, 2001; Findeisen, Diehl, Uslu, Schwarzkopf, Allgöwer, Bock, Schlöder and Gilles, 2002; Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2002; Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2003). The efficiency of the outlined method is underpinned considering the control of a high-purity distillation column.

In **Chapter 4** the stabilization of continuous time systems using sampled-data open-loop feedback is considered. In particular, general stability conditions for sampled-data open-loop feedback are derived, which are an expansion of the results presented in (Findeisen and Allgöwer, 2004d). The results are motivated by ideas utilized in sampled-data open-loop NMPC. However, they are not limited to NMPC. They are rather applicable to a wide class of sampled-data open-loop feedbacks. Notably, the derived stability results allow for varying recalculation intervals and the consideration of constraints on inputs and states. The results are exemplified considering a generalized stability result for sampled-data open-loop NMPC and a new approach for deriving stabilizing sampled-data open-loop feedbacks based on stabilizing instantaneous feedback laws. Furthermore, the practically important question how delays can be considered in sampled-data open-loop feedback is examined. Based on the results presented in (Findeisen and Allgöwer, 2004a; Findeisen and Allgöwer, 2004d), two new delay compensation techniques for the compensation of measurement and computational delays retaining nominal stability are presented. The derived results are exemplified considering the control of a continuous stirred tank reactor.

The question whether sampled-data open-loop feedback possesses inherent robustness properties is considered in **Chapter 5**. It is shown that, under certain Lipschitz conditions, sampled-data open-loop feedbacks possess inherent robustness properties with respect to additive disturbances in the differential equations, to input disturbances, and to measurement uncertainties. The derived robustness results have a series of direct implications. With respect to NMPC they underpin the intuition that small errors in the optimal input trajectory, e.g. resulting from an approximate numerical solution, can be tolerated. The results are an extension of the results presented in (Findeisen et al., 2003e; Findeisen et al., 2003c) for sampled-data open-loop NMPC.

The inherent robustness properties of sampled-data open-loop feedbacks paves the way to sampled-data open-loop output-feedback schemes that achieve semi-regional practical stability (**Chapter 6**). For a broad class of sampled-data open-loop feedback controllers, conditions on the facilitated state observer are derived guaranteeing that the closed-loop is semi-regional practically stable. It is shown that sufficient conditions to achieve semi-regional practical stability are that the used observer achieves a sufficiently fast convergence of the estimation error, and that the value function of the used sampled-data open-loop feedback is locally Lipschitz. The condition on the observer error convergence is in general very stringent. However, a series of observers such as high-gain observers, moving horizon observers and observers with finite convergence time do satisfy it. The results presented are generalizations of the results for the NMPC case as presented in (Imsland, Findeisen, Bullinger, Allgöwer and Foss, 2003; Findeisen et al., 2003b; Findeisen et al., 2003d; Findeisen et al., 2003c). The resulting performance and stability of the closed-loop are discussed considering two example systems, the control of a pendulum car system and the control of a mixed-culture bioreactor.

Chapter 7 summarizes the achieved results and provides an outlook on possible future research directions and open questions in the area of sampled-data open-loop feedback, especially NMPC.

Chapter 2

A Brief Review of Nonlinear Model Predictive Control

In this chapter we review the basic principle of NMPC for continuous time systems, outline the key advantages and disadvantages of this control approach, and discuss the differences between sampled-data open-loop NMPC and instantaneous NMPC. This chapter does not provide a complete review of NMPC; it is rather intended to provide the basis for the following chapters. For more comprehensive reviews the reader is referred to (Mayne et al., 2000; De Nicolao et al., 2000; Allgöwer et al., 1999; Chen and Allgöwer, 1998a; Rawlings, 2000; Allgöwer et al., 2004; Findeisen and Allgöwer, 2001; Findeisen et al., 2003d). Especially, we do not consider the stabilization of discrete time systems using NMPC. Detailed discussion in this respect can be found in (Mayne et al., 2000; De Nicolao et al., 2000; Rawlings, 2000; Allgöwer et al., 1999; Rawlings et al., 1994).

2.1 Basic Principle of Model Predictive Control

The input applied in model predictive control is given by the repeated solution of a (finite) horizon open-loop optimal control problem subject to the system dynamics, input and state constraints. Based on measurements obtained at a time t, the controller predicts the dynamic behavior of the system over the so called control/prediction horizon T_p and determines the input such that an open-loop performance objective is minimized[1]. Under the assumption that the prediction horizon spans to infinity and that there are no disturbances and no model plant mismatch, one could apply the resulting input open-loop to the system and achieve (under certain assumptions) convergence of the system states to the origin. However, due to external disturbances, model plant mismatch and the use of finite prediction horizons the actual predicted state and the true system state differ. Thus, to counteract this deviation and to suppress the disturbances it is necessary to incorporate feedback. In model predictive control this is achieved by applying the obtained optimal open-loop input only until the recalculation

[1]For simplicity of presentation we assume that the prediction and control horizon, as sometimes considered (Morari and Lee, 1999; Qin and Badgwell, 2000; Magni, De Nicolao and Scattolini, 2001b), coincide.

time t_r, at which the whole process – prediction and optimization – is repeated (compare Figure 2.1), thus moving the prediction horizon forward. The whole procedure can be summarized as follows:

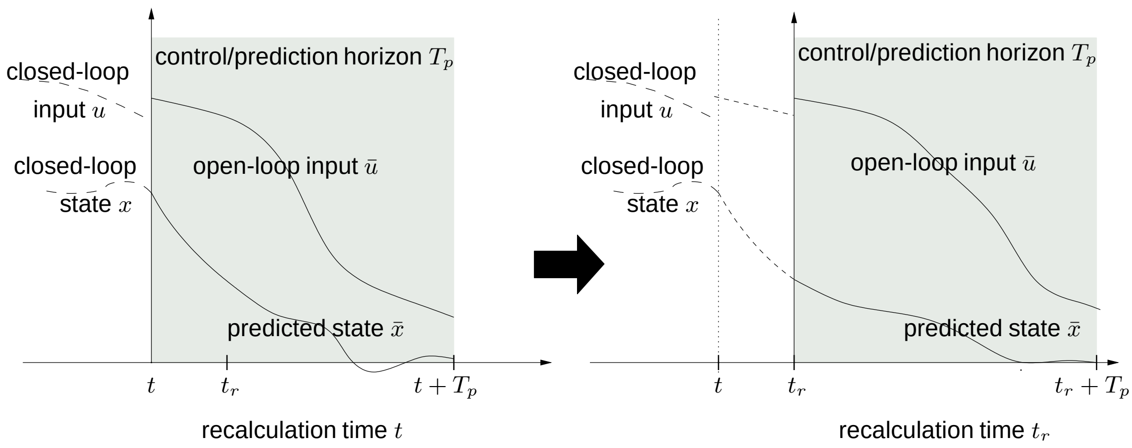


Figure 2.1: Principle of model predictive control.

1. Obtain estimates of the *current* state of the system.
2. Obtain an admissible optimal input by *minimizing* the desired *cost function* over the prediction horizon using the system model and the current state estimate for prediction.
3. Implement the obtained optimal input until the next sampling instant.
4. Continue with 1.

2.2 Basic Mathematical Formulation of NMPC

We consider the following nonlinear system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \in \mathcal{X}_0 \tag{2.1}$$

subject to the input and state constraints

$$u(t) \in \mathcal{U}, \quad x(t) \in \mathcal{X}, \quad \forall t \geq 0, \tag{2.2}$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ is the system state, $u(t) \in \mathcal{U} \subset \mathbb{R}^m$ is the input applied to the system. Here the set of possible inputs is denoted by $\mathcal{U}$, the set of feasible states is denoted by $\mathcal{X}$, and the set of considered initial conditions is denoted by $\mathcal{X}_0 \subseteq \mathbb{R}^n$. With respect f we assume that $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ it is continuous, satisfies $f(0, 0) = 0$, and is locally Lipschitz in x. With respect to the sets $\mathcal{X}$, $\mathcal{X}_0$, $\mathcal{U}$ we assume that $\mathcal{U} \subset \mathbb{R}^m$ is compact, $\mathcal{X} \subseteq \mathbb{R}^n$ is simply connected, $\mathcal{X}_0 \subseteq \mathcal{X}$, and $(0, 0) \in \mathcal{X} \times \mathcal{U}$. Thus, the origin is a stationary point for (2.1).We furthermore denote the solution of (2.1) (if it exists) starting at a time t_1 from a state $x(t_1)$, applying an input $u : [t_1, t_2] \to \mathbb{R}^m$ by $x(\tau; x(t_1), u(\cdot))$, $\tau \in [t_1, t_2]$, i.e.

$$x(\tau; x(t_1), u(\cdot)) = x(t_1) + \int_{t_1}^{\tau} f(x(s), u(s))ds \quad \forall \tau \in [t_1, t_2]. \tag{2.3}$$

In NMPC the feedback is defined via the repeated solution of an open-loop optimal control problem. The open-loop optimal control problem to solve is often formulated as

$$\min_{\bar{u}(\cdot)} \; J(\bar{x}(\cdot), \bar{u}(\cdot)) \tag{2.4a}$$

$$\text{subject to:} \quad \dot{\bar{x}}(\tau) = f(\bar{x}(\tau), \bar{u}(\tau)), \quad \bar{x}(t) = x(t), \tag{2.4b}$$

$$\bar{u}(\tau) \in \mathcal{U}, \quad \tau \in [t, t + T_p] \tag{2.4c}$$

$$\bar{x}(\tau) \in \mathcal{X}, \quad \tau \in [t, t + T_p], \tag{2.4d}$$

$$\bar{x}(t + T_p) \in \mathcal{E} \tag{2.4e}$$

where the cost functional J is defined over the prediction horizon T_p

$$J(x(\cdot), u(\cdot)) = \int_{t}^{t+T_p} F(x(\tau), u(\tau))d\tau + E(x(t + T_p)) \tag{2.4f}$$

in terms of the stage cost F and a terminal penalty term E which specify the desired performance.

The bar denotes internal controller variables. The distinction between the real system variables and the variables in the controller is necessary, since even in the nominal case the predicted values are not the same as the actual closed-loop values. This difference is due to the re-optimization (over the moving finite horizon T_p).

The stage cost F often arises from economical, ecological, or safety considerations. Often a quadratic stage cost function is used, i.e. $F(x, u) = x^T Q x + u^T R u$, with weighting matrices $Q > 0$ and $R \geq 0$. The terminal penalty term E together with the terminal region constraint (2.4e), where $\mathcal{E}$ denotes the so-called terminal set around the origin, are typically used to enforce stability of the closed-loop, see Section 2.5.1 and Chapter 4. The terminal penalty term E typically penalizes the distance of the last predicted state to the origin. With respect to the stage cost F, we assume that $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuous, satisfies $F(0, 0) = 0$, and is lower bounded by a class $\mathcal{K}$-function[2] α_F, i.e. $\alpha_F(x) \leq F(x, u)$.

In the following, optimal solutions of the dynamic optimization problem (2.4) are denoted by a superscript $\star$. For example the optimal input (assuming that it exists) that minimizes (2.4) starting from $x(t)$ is denoted by $\bar{u}^\star(\cdot; x(t)) : [t, t + T_p] \to \mathbb{R}^m$. The input applied to the system is based on the optimal input $u^\star$, as explained in the next section.

The optimal cost of (2.4) as a function of the state is referred to as *value function* and is given by

$$V(x(t)) = J(x(\cdot; x(t), \bar{u}^\star(\cdot; x(t))), \bar{u}^\star(\cdot; x(t))). \tag{2.5}$$

[2] A continuous function $\alpha : [0, \infty) \to [0, \infty)$ is a class $\mathcal{K}$ function, if it is strictly increasing and $\alpha(0) = 0$.

The value function plays a crucial role in the stability considerations of NMPC.

Depending on how “often” the optimal control problem (2.4) is recalculated, different versions of NMPC exist. If the open-loop is solved at all time instants we refer to it as *instantaneous NMPC*. If the dynamic optimization is solved only at disjoint recalculation instants and if the resulting optimal control signal is implemented open-loop in between, the resulting scheme is referred to as *sampled-data open-loop NMPC*. Both schemes have different theoretical as well as practical characteristics.

2.2.1 Instantaneous NMPC

We refer to NMPC schemes that apply at *every time instant* the optimal control problem (2.4) as *instantaneous NMPC*:

Definition 2.1 (Instantaneous NMPC)
The applied input in instantaneous NMPC is given by

$$u(x(t)) = \bar{u}^{\star}(t; x(t)), \tag{2.6}$$

leading to the nominal closed-loop system

$$\dot{x}(t) = f\left(x(t), \bar{u}^{\star}(t; x(t))\right). \tag{2.7}$$

Instantaneous NMPC schemes have the advantage that the system (2.7) is purely continuous time. Thus, standard Lyapunov stability definitions and standard stability result can be utilized. However, also certain problems arise. For example, if the open-loop optimization provides a discontinuous input in terms of the state, the solution of the differential equation might not be defined in the classical Carathéodory sense, since the right-hand side of the differential equation can be discontinuous and switch infinitely fast near “singular” points. More details can be found in (Fontes, 2003; Fontes, 2000b; Michalska and Vinter, 1994). One advantage of instantaneous NMPC is that under certain regularity and continuity assumptions it inherits well known stability properties of optimal control, i.e. it possesses a sector gain margin of $(1/2, \infty)$ to static input nonlinearities similar to the linear quadratic regulator (Chen and Shaw, 1982; Magni and Sepulchre, 1997). The inherent robustness of instantaneous NMPC can for example be used to derive an output-feedback instantaneous NMPC scheme using high-gain observers (Imsland, Findeisen, Bullinger, Allgöwer and Foss, 2003).

While instantaneous NMPC is theoretically appealing, often it can not be applied in practice, since the numerical solution of the corresponding optimal control problem requires some non negligible computation time. While in principle short “delays” (and optimization errors) can be tolerated (Mayne and Michalska, 1990), the longer the necessary computation time, the more undesirable instantaneous NMPC becomes.

2.2.2 Sampled-data Open-loop NMPC

In sampled-data open-loop NMPC the optimal control problem (2.4) is only solved at fixed recalculation instants. Between the recalculation instants the optimal input is applied open-loop. We denote the recalculation instants by t_i. Often the time between the recalculations is assumed to be constant. However, for practical reasons it might be necessary to consider varying recalculation times. For example the computation time available for the solution of the open-loop optimal control problem, as well as the availability of state information are often determined externally and might vary. Thus, we consider in this thesis that the recalculation instants t_i are given by a partition π of the time axis[3].

Definition 2.2 (Partition)
A partition is a series $\pi = (t_i)$, $i \in \mathbb{N}$ of (finite) positive real numbers such that $t_0 = 0$, $t_i < t_{i+1}$ and $t_i \to \infty$ for $i \to \infty$. Furthermore, $\bar{\pi} = \sup_{i\in\mathbb{N}}(t_{i+1} - t_i)$ denotes the upper diameter (longest recalculation time) of π and $\underline{\pi} = \inf_{i\in\mathbb{N}}(t_{i+1} - t_i)$ denotes the lower diameter (shortest recalculation time) of π.

Whenever t and t_i appear together, t_i should be taken as the closest previous recalculation instant with $t_i \leq t$. Whenever t_i and t_{i+k}, $k \in \mathbb{N}$ appear together, t_{i+k} denotes the k^{th} successor element of t_i in the series π. For all considerations in this thesis we assume that the upper and lower diameter of π are finite. For practical applications this assumption is always satisfied.

For sake of brevity we denote in the following the time between two recalculation instants as recalculation time:

Definition 2.3 (Recalculation time δ_i)
The recalculation time corresponding to any $t_i \in \pi$ is defined as

$$\delta_i = t_{i+1} - t_i. \tag{2.8}$$

Whereas in instantaneous NMPC the optimal control problem is solved at all times, in sampled-data open-loop NMPC it is only solved at the recalculation instants.

Definition 2.4 (Sampled-data Open-loop NMPC)
The applied input in sampled-data open-loop NMPC is given by repeated solutions of the optimal control problem (2.4)

$$u(t) = \bar{u}^\star(\tau; x(t_i)). \tag{2.9}$$

Furthermore, the nominal closed-loop system *is given by*

$$\dot{x}(t) = f\left(x(t), \bar{u}^\star(t; x(t_i))\right). \tag{2.10}$$

Thus, only at the recalculation instants t_i the applied open-loop u is recalculated.

2.3 Properties, Advantages and Drawbacks of NMPC

Ideally one would like to use an infinite prediction horizon, i.e. T_p in (2.4f) set to ∞, since this would in the nominal case allow to minimize the overall cost. However, solving a nonlinear optimal

[3] The notation used is similar to the one used in (Clarke et al., 1997; Marchand and Alamir, 2000).

control problem over an infinite horizon is often computationally not feasible. Thus typically a finite prediction horizon is used. In this case the actual closed-loop input and state trajectories differ from the predicted open-loop trajectories, even if no model plant mismatch and no disturbances are present. This can be explained considering somebody hiking in the mountains without a map. The goal of the hiker is to take the shortest route to his goal. Since he is not able to see “infinitely” far (or up to his goal), the only thing he can do is to plan a certain route based on the current information (skyline/horizon) and then follow this route. After some time the hiker reevaluates his route based on the fact that he might be able to see further. The new route obtained might be significantly different from the previous route and he will change his route, even though he has not yet reached the end of the previous considered route.

Basically, the same approach is employed in a finite horizon NMPC strategy. At a recalculation instant the future is only predicted over the prediction horizon. At the next recalculation instant the prediction horizon moves further, thus allowing to obtain more information and re-planning. This is depicted in Figure 2.2, where the system can only move inside the shaded area as state constraints are present. The difference between the predicted trajectories and the closed-loop trajectories has two

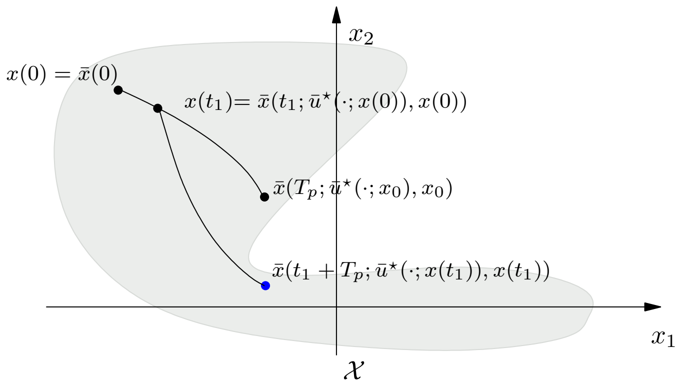


Figure 2.2: Mismatch between open-loop prediction and closed-loop behavior.

immediate consequences.

Firstly, the actual goal of computing a feedback minimizing the performance objective over the *infinite horizon* of the closed-loop is not achieved. In general, it is by no means true that a repeated minimization over a moving *finite horizon objective* leads to an optimal solution for the corresponding infinite horizon problem. The solutions will often even differ significantly if a short finite horizon is chosen.

Secondly, if the predicted and the actual trajectory differ, there is no guarantee that the closed-loop system will be stable. It is indeed easy to construct examples for which the closed-loop becomes unstable if a short finite horizon is chosen, see for example (Bitmead et al., 1990; Muske and Rawlings, 1993). Hence, when using finite prediction horizons the problem must be modified to guarantee stability.

The basic overall structure of an NMPC control loop is shown in Figure 2.3. Based on the applied input and the measured outputs a state estimate is obtained. This estimate is fed into the NMPC

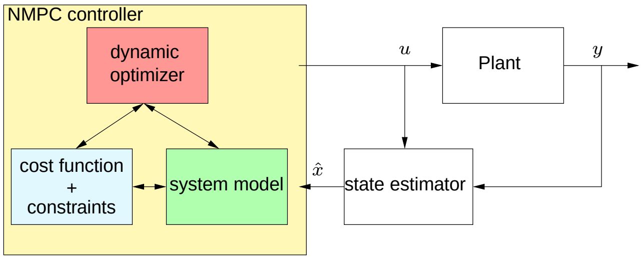


Figure 2.3: Basic NMPC control loop.

controller which computes a new input that is applied to the system. Briefly the key characteristics and properties of NMPC are:

- NMPC allows the direct use of nonlinear models for prediction.
- NMPC allows the explicit consideration of state and input constraints.
- In NMPC a specified time domain performance criteria is minimized on-line.
- In NMPC the predicted behavior is in general different from the closed-loop behavior.
- The implementation of NMPC requires the on-line solution of an open-loop optimal control problem.
- To perform the prediction the system states must be measured or estimated.

Remark 2.1 *In this work we mainly focus on NMPC for the stabilization of time-invariant nonlinear systems. However, NMPC is also applicable to other classes of systems, i.e. discrete time systems, delay systems, time-varying systems, and distributed parameter systems, for more details see for example (Mayne et al., 2000; De Nicolao et al., 2000; Allgöwer et al., 1999). Furthermore, NMPC is also well suited for tracking problems or problems where an optimal transfer between steady-states must be performed, see (Magni, De Nicolao and Scattolini, 2001a; Michalska, 1996; Findeisen, Chen and Allgöwer, 2000; Findeisen and Allgöwer, 2000b; Tenny et al., 2002; Wan and Kothare, 2003a).*

Many of the mentioned properties can be seen as advantages as well as drawbacks of NMPC. The possibility to directly use a nonlinear model is advantageous if a detailed first principles model is available. In this case often the performance of the closed-loop can be increased significantly without much tuning. Nowadays first principle models of a plant are often derived before a plant is built. Especially the process industry has a strong desire to use (rather) detailed models from the first design up to the operation of the plant for reasons of consistency and cost minimization. On the other side, if no first principle model is available, it is often dfifficult to obtain a good nonlinear model based on identification techniques. In this case it might be better to apply other control strategies.

2.4 Numerical Aspects of NMPC

Predictive control circumvents the solution of the Hamilton-Jacobi-Bellman equation by solving the open-loop optimal control problem at every sampling instant only for the currently (measured) system state. Nevertheless, the application of NMPC requires the sufficiently fast on-line solution of an optimal control problem. Thus, one important precondition for the applicability of NMPC is the availability of reliable and efficient numerical dynamic optimization algorithms for Problem (2.4). Solving (2.4) numerically efficient and fast is, however, not a trivial task and has attracted much research interest in recent years (see e.g. (Mayne, 1995; Wright, 1996; Bartlett et al., 2000; Tenny and Rawlings, 2001; Biegler, 2000; Li and Biegler, 1989; de Oliveira and Biegler, 1995; Martinsen et al., 2002; Biegler and Rawlings, 1991; Mahadevan and Doyle III, 2003; Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2002; Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2003; Diehl, Findeisen, Nagy, Bock, Schlöder and Allgöwer, 2002)). Typically so called direct solution methods (Binder et al., 2001; Biegler and Rawlings, 1991; Pytlak, 1999) are used, i.e. the original infinite dimensional problem is approximated by a finite dimensional one discretizing the input (and also possibly the state). Further details on the efficient solution of the optimal control problem (2.4) are provided in Chapter 3.

2.5 System Theoretical Aspects of NMPC

This section briefly reviews and discusses theoretical aspects of NMPC. Besides the question of nominal stability of the closed-loop, remarks on robust NMPC strategies as well as the output-feedback problem are given.

2.5.1 Nominal Stability of NMPC

One elementary question in NMPC is whether a finite horizon NMPC strategy does guarantee stability of the closed-loop. While a finite prediction and control horizon is desirable from an implementation point of view, the difference between the predicted state trajectory and the resulting closed-loop behavior can lead to instability.

The most intuitive way to achieve stability/convergence to the origin is to use an infinite horizon cost, i.e. T_p in Problem 1 is set to ∞. To exemplify the basic ideas behind an NMPC stability proof we briefly outline how stability for the instantaneous case can be established. Detailed derivations for this case can be found in (Mayne and Michalska, 1990; Mayne et al., 2000) for the continuous time case, and in (Keerthi and Gilbert, 1988; Keerthi and Gilbert, 1985) for the discrete time case.

In infinite horizon NMPC, the cost function of the open-loop optimal control problem (2.4) is given by

$$J(x(\cdot),u(\cdot)) = \int_t^{\infty} F(x(\tau),u(\tau))d\tau. \tag{2.11}$$

Stability of Infinite Horizon Instantaneous NMPC: Nearly all stability proofs of instantaneous NMPC schemes use the value function as a Lyapunov function, i.e. they establish that

$$\frac{\partial V}{\partial x}(x)f(x,u(x(t)) \leq -\alpha(\|x\|), \tag{2.12}$$

where α is a $\mathcal{K}$ function. First note that in the nominal case with an infinite horizon due to the principle of optimality (Bellman, 1957) the open and the closed-loop state and input trajectories coincide (end pieces of optimal trajectories are optimal), i.e.

$$x(t) = x^\star(t; x_0, \bar{u}^\star(t_0; x_0)), \quad u(t) = \bar{u}^\star(t_0; x_0).$$

Thus, if the open-loop optimal control problem is feasible at t_0, it is also feasible afterwards. It furthermore follows that

$$V(x(t)) = V(x_0) - \int_{t_0}^{t} F(x(\tau; x_0, \bar{u}^\star(\tau; x_0)), \bar{u}^\star(\tau; x_0))d\tau. \tag{2.13}$$

Under the simplifying assumption that $V(x)$ is C^1, and that the level sets of V are compact differentiating (2.13) with respect to t leads to

$$\frac{\partial V}{\partial x}(x(t))f(x(t), \bar{u}^\star(t; x(t))) \leq -F(x(t), \bar{u}^\star(t; x(t))).$$

Thus, assuming that F is lower bounded by a $\mathcal{K}$ function, V is a Lyapunov function and it is established that the closed-loop is asymptotically stable.

In the following we review existing instantaneous and sampled-data NMPC schemes that guarantee stability and outline some of the differences between them.

2.5.1.1 Stabilizing Instantaneous NMPC Schemes

One of the simplest approaches leading to stability in the case of a finite prediction horizon is to add a so called zero terminal constraint of the form

$$\bar{x}(t + T_p) = 0 \tag{2.14}$$

to the open-loop optimal control problem (Chen and Shaw, 1982; Mayne and Michalska, 1990; Keerthi and Gilbert, 1988; Meadows et al., 1995). This corresponds to shrinking the set $\mathcal{E}$ of (2.4) to zero. This allows, under certain regularity assumptions, to establish asymptotic stability. The feasibility at a specific time follows from the feasibility before, since one can complement the old input by a zero input at the end. In comparison to (2.13) now only an inequality holds, since the complemented input is feasible, but not optimal:

$$V(x(t)) \leq V(x_0) - \int_{t_0}^{t} F(x(\tau; x_0, \bar{u}^\star(\tau; x_0)), \bar{u}^\star(\tau; x_0))d\tau \quad \forall t \in [t_0, t_0 + T_p]. \tag{2.15}$$

This argument holds for all t_0 and t, thus V is a suitable Lyapunov function candidate if additional regularity assumptions are imposed, which are mainly necessary to guarantee that V is continuously differentiable. The regularity assumptions can be relaxed, as shown in (Michalska and Mayne, 1991), merely implying that the value function is locally Lipschitz continuous. In (Michalska and Vinter, 1994; Michalska, 1995) this result is further expanded to the case of time varying systems with state constraints, and in (Michalska, 1996) to the tracking problem.

One disadvantage of a zero terminal constraint is that the predicted system state is forced to reach the origin in finite time. This leads to feasibility problems for short prediction/control horizon lengths, i.e. small regions of attraction. Furthermore, from a computational point of view, an exact satisfaction of a zero terminal equality constraint does require in general an infinite number of iterations in the optimization and is thus not desirable. The main advantages of a zero terminal constraint are the straightforward application and the conceptual simplicity.

Approaches avoiding a zero terminal constraint have been first proposed for sampled-data NMPC, as discussed in the next section. In general, the terminal region constraint (2.4e)

$$\bar{x}(t+T_p) \in \mathcal{E} \tag{2.16}$$

and/or the terminal penalty $E(x)$ in the cost function (2.4f) are used to enforce stability. Basically in these approaches the terminal cost E is assumed to be a F-conform control Lyapunov function for the system in the terminal region $\mathcal{E}$, enforcing a decrease in the value function. The terminal region constraint is added to enforce that if the open-loop optimal control problem is feasible once, that it will remain feasible, and to allow establishing the decrease using E.

The work in (Mayne et al., 2000) presents a rather general framework for stabilizing instantaneous NMPC schemes. This framework is summarized in the following theorem.

Theorem 2.1 (Stability of Instantaneous NMPC)
Suppose that $\mathcal{E}$ and E are such that

(a) the value function $V(x)$ is continuously differentiable as a function of x.

(b) E is C^1 with $E(0)=0$, $\mathcal{E} \subseteq \mathcal{X}$ is closed and connected with the origin contained in $\mathcal{E}$ and there exists a continuous local controller $k(x)$ that renders $\mathcal{E}$ invariant, satisfies the input constraints, i.e. for any $x \in \mathcal{E},\ k(x) \in \mathcal{U}$, and guarantees that the following holds:

$$\frac{\partial E}{\partial x} f(x,k(x)) + F(x,k(x)) \leq 0, \quad \forall x \in \mathcal{E}.$$

(c) the NMPC open-loop optimal control problem has a feasible solution for t_0.

Then the nominal closed-loop system defined by (2.1), (2.4) *and* (2.6) *is asymptotically stable. Furthermore, the region of attraction is given by the set of states for which the open-loop optimal control problem has a feasible solution.*

This framework includes, under further regularity assumptions, the zero terminal constraint NMPC approach ($\mathcal{E}=0$) and the infinite horizon NMPC approach ($T_p=\infty$). The key points are the decrease condition implied by assumption (b) and the invariance of the terminal region $\mathcal{E}$ under the local control law, implying feasibility if an initial feasible solution exists.

In (Michalska, 1997; Gyurkovics, 1998) it is shown that adding a terminal region constraint of the form (2.4e) can be avoided without jeopardizing asymptotic stability. The result in (Michalska, 1997) establishes that if the weight on the terminal state $\bar{x}(t+T_p)$ is sufficiently large that the closed-loop is stable. This result is in agreement with similar observations in the linear MPC case (Bitmead et al., 1990). The result presented in (Gyurkovics, 1998) is based on a generalization of the so-called Fake Riccati Equation techniques. Basically it is shown that if the terminal penalty term is chosen such that the Hamiltonian function of the system is "negative" that the closed-loop is asymptotically stable.

2.5.1.2 Stabilizing Sampled-data Open-loop NMPC Schemes

For sampled-data open-loop NMPC in principle similar approaches that guarantee stability as in the instantaneous case exist. However, additionally the behavior of the closed-loop in between the recalculation instants must be taken into account. The main advantage of a sampled-data open-loop NMPC implementation is that no differentiability assumption on the value function is necessary, since the open-loop input is applied over a "finite" time, see e.g. (Fontes, 2000b; Fontes and Magni, 2003) and the generalized results presented in Chapter 4.

In principle five different approaches for achieving stability of sampled-data open-loop NMPC can be distinguished: NMPC schemes using an infinite prediction horizon, NMPC schemes that switch to a local controller to achieve asymptotic stability near the origin, NMPC schemes where the convergence is enforced by a terminal region constraint and a terminal penalty term, NMPC schemes using control Lyapunov functions to establish convergence, and NMPC schemes enforcing stability by adding a direct contraction condition/decrease condition to the optimal control problem.

Stability via a zero terminal constraint:
Similar to instantaneous NMPC stability of the closed-loop can be enforced by adding a zero terminal constraint of the format (2.14) to the open-loop optimal control problem. The convergence of such a scheme follows from the results presented in (Fontes, 2000b).

Dual-mode control:
One of the first sampled-data NMPC approaches avoiding an infinite horizon or a zero terminal constraint is the so called dual-mode NMPC approach (Michalska and Mayne, 1993). Dual-mode is based on the assumption that a local (linear) controller is available for the nonlinear system. Based on this local linear controller a terminal region and a quadratic terminal penalty term are added to the open-loop optimal control problem similar to E and $\mathcal{E}$ such that: 1.) the terminal region is invariant under the local control law, 2.) the terminal penalty term E enforces a decrease in the value function. Furthermore the prediction horizon is considered as additional degree of freedom in the optimization. The terminal penalty term E can be seen as an approximation of the infinite horizon cost inside of the terminal region $\mathcal{E}$ under the local linear control law. Note, that dual-mode control is not a "pure" NMPC controller, since the open-loop optimal control problem is only repeatedly solved until the system state enters the terminal set $\mathcal{E}$, which is achieved in finite time. Once the system state is inside $\mathcal{E}$ the control is switched to the local control law $u = Kx$, thus the name dual-mode NMPC. Thus, the

local control is utilized to establish asymptotic stability while the NMPC feedback is used to increase the region of attraction of the local control law.

Control Lyapunov function approaches:
In the case that E is a global control Lyapunov function for the system, the terminal region constraint $\bar{x}(t+T_p) \in \mathcal{E}$ is actually not necessary. Even if the control Lyapunov function is not globally valid, convergence to the origin can be achieved (Jadbabaie et al., 2001; Ito and Kunisch, 2002; Sznaier et al., 2003; Sznaier and Cloutier, 2001) and it can be established that for increasing prediction horizon length the region of attraction of the infinite horizon NMPC controller is recovered (Jadbabaie et al., 2001). For all these approaches it is in general rather difficult to consider constraints on the inputs and states, since it is in general rather difficult to obtain suitable control Lyapunov functions.

Convergence by enforced contraction:
Besides the approaches presented so far it is also possible to enforce the stability of NMPC directly, as in contractive NMPC (de Oliveira Kothare and Morari, 2000; Yang and Polak, 1993). In these approaches an explicit contraction constraint of the form

$$\|\bar{x}(t_{i+1})\| \leq \beta \|x(t_i)\|, \quad \beta \in (0,1),$$

is added to the open-loop optimal control problem. This constraint directly enforces the contraction of the state at the recalculation instants. The main problem with respect to this approach is that the feasibility at one time instant does not necessarily imply the feasibility at the next recalculation instant, thus strict assumptions on the system must be made. Furthermore, assumptions on the well behavedness of the system in between recalculation instants are necessary.

A "mixture" of enforced contraction and the control Lyapunov function approach is considered in (Primbs et al., 2000). In this work a direct control Lyapunov function decreases in the cost function along solution trajectories with a required decrease of the control Lyapunov function value at the end of the prediction horizon is used. Thus, the degree of freedom left in the control Lyapunov function is utilized in NMPC to minimize the considered objective function. In the limit for $T_p \to 0$ this approach converges to the min-norm controller, while for $T_p \to \infty$ the approach converges to an infinite horizon optimal control law.

Unified conditions for convergence:
Besides the outlined approaches there exist a series of approaches (Michalska and Mayne, 1993; Chen and Allgöwer, 1998b; Chen and Allgöwer, 1998a; Chen et al., 2000; Magni and Scattolini, 2002) that are based on the consideration of a (virtual) local control law that is able to stabilize the system inside of the terminal region and where the terminal penalty E provides an upper bound on the optimal infinite horizon cost. Similar to (Mayne et al., 2000) for the instantaneous case, (Fontes, 2000b) proposes a unifying frame for sampled-data NMPC. This frame even allows considering the stabilization of systems which can be only stabilized by discontinuous control. The following theorem

covers most of the existing stability results. It establishes conditions for the convergence of the closed-loop states under sampled-data NMPC. It is a slight modification of the results given in (Fontes, 2000b; Chen and Allgöwer, 1998a; Chen, 1997).

Theorem 2.2 (Convergence of sampled-data open-loop NMPC)
Suppose

(a) the terminal region $\mathcal{E} \subseteq \mathcal{X}$ *is closed with* $0 \in \mathcal{E}$ *and the terminal penalty* $E(x) \in C^1$ *is positive semi-definite.*

(b) $\forall x(0) \in \mathcal{E}$ *there exists an (admissible) input* $u_{\mathcal{E}} : [0, \bar{\pi}] \rightarrow \mathcal{U}$ *such that*

$$\frac{\partial E}{\partial x} f(x(\tau), u_{\mathcal{E}}(\tau)) + F(x(\tau), u_{\mathcal{E}}(\tau)) \leq 0, \quad \text{and } x(\tau; x(0), u_{\mathcal{E}}(\cdot)) \in \mathcal{E} \quad \forall \tau \in [0, \bar{\pi}]. \quad (2.17)$$

(c) the NMPC open-loop optimal control problem has a feasible solution for t_0*.*

Then for the closed-loop system defined by (2.1), (2.4) *and* (2.9), $x(t) \rightarrow 0$ *for* $t \rightarrow \infty$*. Furthermore, the region of attraction is given by the set of states for which the open-loop optimal control problem has a feasible solution.*

The proof of this theorem can be derived as a special case of the results presented in Chapter 4.

Remark 2.2 *With respect to all presented approaches it should be noted that it is not strictly necessary to find a global minimum of the optimal control problem at every sampling instant. Instead, the optimality can be replaced by requiring that the value function is decreasing sufficiently from recalculation instant to recalculation instant while guaranteeing feasibility at the next recalculation instant. Thus, feasibility and a decrease in the value function can be seen as leading to closed-loop stability, i.e."feasibility implies stability" (Scokaert et al., 1999; Michalska and Mayne, 1993; Chen and Allgöwer, 1998b; Findeisen and Rawlings, 1997; Findeisen, 1997; Jadbabaie et al., 2001).*

Remark 2.3 *Various ways to determine a suitable terminal penalty term and terminal region exist. Examples are the use of a control Lyapunov function as terminal penalty* E *(Jadbabaie et al., 2001) or the use of a local nonlinear or linear control law to determine a suitable terminal penalty* E *and a terminal region* $\mathcal{E}$ *(Michalska and Mayne, 1993; Chen and Allgöwer, 1998b; Chen and Allgöwer, 1998a; Chen et al., 2000; Magni and Scattolini, 2002).*

Remark 2.4 *The key advantage of using a terminal penalty* E *and terminal region constraints is the fact that the open-loop optimal control problem is relaxed, thus leading to an often significantly decreased time required for the numerical solution of the open-loop optimal control problem, see for example (Chen and Allgöwer, 1998b; Findeisen, Nagy, Diehl, Allgöwer, Bock and Schlöder, 2001). This issue is further discussed in Chapter 3. Furthermore, it is possible to take constraints on the states and inputs into account, which is typically a problem for approaches based on control Lyapunov function considerations. In comparison to zero terminal constraint NMPC, the performance of the closed-loop with respect to the "maximum" achievable performance by an infinite horizon NMPC* *scheme can be increased significantly. The terminal penalty term E can be seen as an upper bound of the optimal infinite horizon cost inside of the terminal region $\mathcal{E}$ (Chen and Allgöwer, 1998b). If the local controller used for deriving E does lead to similar performance as an optimal controller with an infinite horizon, similar performance of the finite horizon NMPC scheme and an infinite horizon NMPC scheme can be expected. Thus, the terminal penalty term E can be seen as quasi expanding the prediction horizon to infinity, giving the control scheme in (Chen and Allgöwer, 1998b) its name, quasi-infinite horizon NMPC.*

2.5.2 Robustness and Robust Design of NMPC

The NMPC schemes presented up to now are based on the assumption that the actual system is identical to the model used for prediction, i.e. that no model-plant mismatch or unknown disturbances are present. Clearly, this is very unrealistic for practical applications and the development of an NMPC framework to address robustness issues is of paramount importance. In general, one distinguishes between the inherent robustness properties of NMPC and NMPC schemes taking the uncertainty/disturbances directly into account.

The inherent robustness of NMPC corresponds to the fact that nominal NMPC can cope with uncertainties and disturbances without taking them directly into account. The inherent robustness of NMPC property stems from the close relation of NMPC to optimal control. Results on the inherent robustness of instantaneous NMPC can for example be found in (Magni and Sepulchre, 1997; Chen and Shaw, 1982; Mayne et al., 2000). Discrete time results are given in (Scokaert et al., 1997). Results for specific sampled-data NMPC implementations can be found in (Michalska and Mayne, 1993; Yang and Polak, 1993). In Chapter 5 we expand these results to the general sampled-data open-loop feedback case.

Most robust NMPC schemes taking the uncertainty/disturbance directly into account are based on a min-max formulation. At least three main formulations can be distinguished:

Robust NMPC solving an open-loop min-max problem (Lall and Glover, 1994; Chen et al., 1997; Blauwkamp and Basar, 1999):
In this formulation the standard NMPC setup is kept. However, the cost function takes the worst case uncertainty (or disturbance) out of a set $\mathcal{D}$ into account. Thus, the following min-max problem is solved on-line

$$\min_{\bar{u}(\cdot)} \max_{\Delta \in \mathcal{D}} \int_t^{t+T_p} F(\bar{x}(\tau), \bar{u}(\tau)) d\tau + E(\bar{x}(t+T_p))$$

subject to: $\dot{\bar{x}}(\tau) = f_\Delta\left(\bar{x}(\tau), \bar{u}(\tau)\right), \quad \bar{x}(t) = x(t)$.
Here f_Δ is the system realization including the uncertainty. The resulting open-loop optimization is a min-max problem. Adding stability constraints similar to the nominal case is difficult since no feasible solution might be found at all, as all possible uncertainty/disturbance scenarios have to be considered. One open-loop input signal must lead to stability for a whole class of systems “spanned” by the uncertainty while guaranteeing satisfaction of the stability constraints.

H_∞ **based NMPC** (Magni, Nijmeijer and van der Schaft, 2001; Chen et al., 1997; Magni, De Nicolao, Scattolini and Allgöwer, 2001):
Another possibility is to consider the standard H_∞ problem in a receding horizon framework. The main problem for a practical application of this approach is the prohibitive computation time and the fact that a global optimum of a dynamic min-max problem must be found in order to guarantee robust stability.

Robust NMPC via optimizing a feedback controller used in between the sampling times (Kothare et al., 1996; Magni, De Nicolao, Scattolini and Allgöwer, 2001; Fontes and Magni, 2003):
The open-loop formulation of the robust stabilization problem can be seen as very conservative, since only open-loop control is used during the sampling times, i.e. the disturbances are not directly rejected in between the sampling instants. Instead of optimizing the open-loop input signal directly, one can search for an optimal *feedback controller* that is applied in between the sampling instants, thus introducing instantaneous feedback. In this approach the optimization variables are the design parameter of a "sequence" of control laws $u_i = k_i(x)$ applied in between the sampling instants, i.e. the optimization problem has as optimization variables the parameters of the feedback controllers $\{k_1, \ldots, k_N\}$. This formulation overcomes the conservatism of the first approach, since not one single input signal must overcome all possible disturbances. Nevertheless the solution is often still prohibitively complex.

Summarizing, by now most of the robust NMPC designs are computationally intractable for a practical application. Thus, the analysis of inherent robustness properties of NMPC is of special interest, to at least allow an answer to the question if sufficiently small disturbances can be rejected. This will be considered for sampled-data open-loop NMPC in more detail in Chapter 5.

2.5.3 Output-Feedback and NMPC

One of the key obstacles for the application of NMPC is that at every sampling instant t_i the system state is required for prediction. However, often not all system states are directly accessible, i.e. only the output y is directly available for feedback:

$$y = h(x, u) \tag{2.18}$$

where $y(t) \in \mathbb{R}^p$ are the measured outputs and where $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ maps the state to the output. To overcome this problem one typically employs a state observer for the reconstruction of the states. In principle, instead of the optimal feedback the following feedback, based on the certainty equivalence principle, is applied:

$$u(t) = \bar{u}^\star(t; \hat{x}(t_i)), \tag{2.19}$$

where $\hat{x}$ denotes a state estimate provided by a state observer. Yet, due to the lack of a general nonlinear separation principle, stability is not guaranteed, even if the state observer and the NMPC controller are both stable.

Various researchers have addressed the question of output-feedback NMPC using observers for state recovery, for a more comprehensive review we refer to (Findeisen et al., 2003d). We restrict the discussion to output-feedback model predictive control schemes relying on state space models for prediction and differentiate between the two output-feedback design approaches as outlined above.

The "certainty equivalence"-method is often used in a somewhat ad-hoc manner in industry (Qin and Badgwell, 2003), e.g. based on the (extended) Kalman filter as a state observer. In the presence of a separation principle, this would be a theoretically sound way to achieve a stabilizing output-feedback scheme. Unfortunately, a general separation principle does not exist for MPC — even in the case of linear models, the separation principle for linear systems is void due to the presence of constraints. Thus, at the outset, nothing can be said about closed loop stability in this case, and it seems natural that one has to restrict the class of systems one considers to obtain results. As an example, (Zheng and Morari, 1995) shows global asymptotic stability for the special case of discrete-time linear open-loop *stable* systems.

For a more general class of nonlinear systems, it can be shown that the properties of the value function as a Lyapunov function gives some robustness of NMPC to "small" estimation errors. For "weakly detectable" discrete-time systems, this is first pointed out in (Scokaert et al., 1997) (see also (Magni et al., 1998; Magni, De Nicolao and Scattolini, 2001a), and an early version in (Muske et al., 1994)). However, these results must be interpreted as "local", in the sense that even though that an approximated region of attraction can be calculated in principle, it is not clear how parameters in the controller or observer must be tuned to influence the size of the region of attraction.

In (de Oliveira Kothare and Morari, 2000), local uniform asymptotic stability of contractive NMPC in combination with a "sampled" EKF state estimator is established.

Non-local results are obtained in (Michalska and Mayne, 1995), where an optimization based moving horizon observer combined with the NMPC scheme proposed in (Michalska and Mayne, 1993) is shown to lead to (semi-global) closed-loop stability. For the results to hold, however, a global optimization problem for the moving horizon observer with an imposed contraction constraint must be solved.

More recently, "regional" separation principle-based approaches have appeared for a wide class of NMPC schemes (Imsland, Findeisen, Bullinger, Allgöwer and Foss, 2003; Findeisen et al., 2003b; Findeisen et al., 2003d). In (Imsland, Findeisen, Bullinger, Allgöwer and Foss, 2003; Findeisen, Imsland, Allgöwer and Foss, 2001; Imsland et al., 2001) it is shown that based on the results of (Atassi and Khalil, 2000; Teel and Praly, 1995), semi-regional practical stability results could be obtained for instantaneous NMPC based on a special class of continuous-time models, using high gain observers for state estimation. In this context, semi-regional practical stability means that for any compact region inside the state-feedback NMPC region of attraction, there exists a sampling time and an observer gain such that for system states starting in this region, the closed loop take the state into any small region containing the origin. The instantaneous result of (Imsland, Findeisen, Bullinger, Allgöwer and Foss, 2003) are generalized in (Findeisen, Imsland, Allgöwer and Foss, 2002; Findeisen et al., 2003b) to sampled-data open-loop feedback. In (Findeisen et al., 2003b) it is specifically pointed out that the results can be seen as a consequence of the inherent robustness NMPC possesses under certain conditions. While all of these results are limited to the use of high-gain observers, the results are generalized in (Findeisen et al., 2003c; Findeisen et al., 2003d) to a wider class of observers and, also, open-loop state-feedback NMPC schemes. Specifically explicit conditions on the estimation error are derived such that the closed-loop is semi-regional practically stable. The condition basically requires that the observer error can be made as small as desired in any desired time. While this condition is in principle very stringent, observer designs exist that achieve the desired properties.

Related results to “regional” separation principle-based approaches appeared recently in (Adetola and Guay, 2003), where for the same system class as considered in (Imsland, Findeisen, Bullinger, Allgöwer and Foss, 2003), semi-regional practical stability results are presented using *discretized* high-gain observers.

In (Wan and Kothare, 2003b) a scheduled state-feedback NMPC scheme is combined with an exponential convergent observer, and regional stability results are established. On a related note, the same authors show in (Wan and Kothare, 2002) how an NMPC controller can be combined with a convergent observer to obtain stability.

In the robust design approach the errors in the state estimate are directly accounted for in the state-feedback predictive controller. For linear systems (Bemporad and Garulli, 2000) introduces a set membership estimator to obtain quantifiable bounds on the estimation error, which are used in a robust constraint-handling predictive controller. The setup of (Bemporad and Garulli, 2000) is taken further in (Chisci and Zappa, 2002), using a more general observer, and considering more effective computational methods. For the same class of systems, (Löfberg, 2002) does joint estimation and control calculation based on a minimax formulation, however without obtaining stability guarantees.

For linear systems with input constraints, the method in (Lee and Kouvaritakis, 2001) obtains stability guarantees through computation of invariant sets for the state vector augmented with the estimation error. In a similar fashion, by constructing invariant sets for the observer error, (Kouvaritakis et al., 2000) adapts the NMPC controller in (Cannon et al., 2001) such that the total closed loop is asymptotically stable.

In Chapter 6 we derive a generalization of the “regional” separation principle-based approaches as presented in (Imsland, Findeisen, Bullinger, Allgöwer and Foss, 2003; Findeisen et al., 2003b; Findeisen et al., 2003d; Findeisen et al., 2003c; Findeisen, Imsland, Allgöwer and Foss, 2002) to the general sampled-data open-loop output-feedback case.

2.6 Summary

In this chapter we reviewed the basics principle behind NMPC and outlined the difference between instantaneous and sampled-data NMPC. Furthermore, we discussed some of the theoretical questions such as stability and robustness, as well as some of the numerical aspects. The chapter lays the notational and conceptual basis for the following chapters.

One of the key problems of sampled-data NMPC is the fact that at every recalculation instant an open-loop optimal control problem must be solved. It has been argued that NMPC will never be applicable to reasonably sized practical control problems, since the solution of the open-loop optimal control problem can not be obtained sufficiently fast. In the next chapter we will address this question. We show that if an NMPC scheme with reduced computational demand and an efficient numerical solution strategy for the resulting dynamic optimization problem are used, then NMPC is real-time applicable to practically relevant, rather large control problems.

Chapter 3

Computational Issues in Sampled-data NMPC

Predictive control circumvents the solution of the Hamilton-Jacobi-Bellman equation by solving the open-loop optimal control problem at every recalculation instant only for the currently (measured) system state. An often intractable problem is replaced by a tractable one. Nevertheless, for a real-time implementation the open-loop optimal control problem (2.4) must be solved efficiently and reliable. According to (Qin and Badgwell, 2000) "Speed and assurance of reliable solution in real-time are major limiting factors in existing applications". Solving (2.4) numerically efficient and fast is, however, not a trivial task and has attracted much research interest in recent years, see e.g. (Mayne, 1995; Wright, 1996; Bartlett et al., 2000; Tenny and Rawlings, 2001; Tenny, 2002; Biegler, 2000; Li and Biegler, 1989; de Oliveira and Biegler, 1995; Martinsen et al., 2002; Biegler and Rawlings, 1991; Mahadevan and Doyle III, 2003; Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2002; Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2003; Diehl, Findeisen, Nagy, Bock, Schlöder and Allgöwer, 2002; Findeisen, Diehl, Uslu, Schwarzkopf, Allgöwer, Bock, Schlöder and Gilles, 2002; Binder et al., 2001; Sistu et al., 1993).

In this chapter we show that a real-time application of NMPC is possible if a "symbiosis" of specially tailored dynamic optimization strategies and NMPC schemes with a reduced computational load are used. For this purpose we first discuss suitable NMPC schemes that facilitate a fast and efficient solution. Then we outline one specific, specially tailored dynamic optimization strategy based on multiple shooting methods, developed in the scope of a computational feasibility study of NMPC (Nagy, Findeisen, Diehl, Allgöwer, Bock, Agachi, Schlöder and Leineweber, 2000; Findeisen, Allgöwer, Diehl, Bock, Schlöder and Nagy, 2000; Diehl, 2002; Diehl, Findeisen, Nagy, Bock, Schlöder and Allgöwer, 2002; Diehl et al., 2001; Findeisen, Nagy, Diehl, Allgöwer, Bock and Schlöder, 2001; Findeisen, Diehl, Bürner, Allgöwer, Bock and Schlöder, 2002; Findeisen, Diehl, Uslu, Schwarzkopf, Allgöwer, Bock, Schlöder and Gilles, 2002; Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2003; Findeisen, Nagy, Diehl, Allgöwer, Bock and Schlöder, 2001; Findeisen and Allgöwer, 2000a). The efficiency of the outlined method is underpinned by means of the control of a high-purity distillation column.

3.1 NMPC Formulations Facilitating Efficient Solutions

Besides an efficient numerical solution of the dynamic optimization problem occurring in NMPC, the real-time applicability also depends strongly on the choice of an NMPC scheme that achieves guaranteed stability and good performance without leading to a high computational demand.

Two approaches reducing the required computational demand of the applied NMPC scheme are shortly discussed in the following.

3.1.1 Use of Short Horizon Lengths and Non-stringent Stability Constraints

Ideally, one seeks to use an infinite control/prediction horizon to achieve good performance and stability of the closed-loop. However, choosing an infinite horizon leads to an infinite dimensional optimization problem, which is not desirable from a computational point of view. One way to achieve stability avoiding an infinite horizon is the use of the so-called zero terminal constraint NMPC scheme (Keerthi and Gilbert, 1988; Mayne and Michalska, 1990), forcing the state at the end of the horizon to the desired steady-state. However, the resulting optimization problem is in general expensive, since a two-point boundary value problem must be solved during optimization. Additionally the control performance may decrease significantly, since the open-loop trajectory has to be forced to reach the set-point in finite time. The NMPC framework (2.4) including a terminal penalty and terminal region constraint allows overcoming this dilemma. Several schemes utilize this frame and require a reduced computational load (Chen and Allgöwer, 1998b; De Nicolao et al., 1996; Jadbabaie et al., 2001; Fontes, 2000b; Primbs et al., 2000; Magni, De Nicolao and Scattolini, 2001b; Sznaier et al., 2003). All of these approaches use a final terminal penalty term E and relax the zero terminal constraint by a terminal region constraint $\bar{x}(t+T_p; x(t)) \in \mathcal{E}$ or even do not require any constraint on the final predicted state at all. The terminal penalty is typically used for the approximation of the infinite horizon cost. The (often) necessary terminal region constraint is in general not very restrictive, i.e. it does not complicate the dynamic optimization problem in a restrictive manner, as for example a zero terminal constraint does. The computational (and performance) advantage of these schemes lies in the fact that shorter horizons can be used, while not jeopardizing performance and stability. The achievable performance of the resulting scheme is close to the infinite horizon one, if the terminal region and a terminal penalty term are chosen suitably. We propose to use this kind of schemes in combination with specially tailored dynamic optimization strategies as outlined in the next section.

In the following we focus on the schemes derived in (Chen and Allgöwer, 1996; Chen and Allgöwer, 1998b; De Nicolao et al., 1996), since they are used in Section 3.4 for the control of a high-purity distillation column. In (De Nicolao et al., 1996) it is proposed to approximate the infinite horizon cost via the terminal penalty term E inside the terminal region $\mathcal{E}$ by utilizing a local control law $u = k(x)$ (often a LQR controller based on the system linearization) which can stabilize the system inside of $\mathcal{E}$. To achieve this, the predicted state at the end of the horizon is forced to lie in the set $\mathcal{E}$. $E(\bar{x}(t+T_p))$ is then obtained by an on-line “integration” of the system up to “infinity” using the local control law, i.e.

$$E(\bar{x}(t+T_p)) = \int_{t+T_p}^{\infty} F(\tilde{x}(\tau), k(\tilde{x}(\tau)))d\tau,$$

where

$$\dot{\tilde{x}} = f(\tilde{x}, k(\tilde{x})), \qquad \tilde{x}(t) = \bar{x}(t).$$

Thus, for every evaluation of the cost the system must be integrated to "infinity". In (De Nicolao et al., 1996) it is shown that for achieving stability it is actually not necessary to integrate to infinity, instead a rather long calculation time into the future is sufficient. Note that the calculation of E based on the integration can be very cheap even on relative long horizons, as for example the step size of the integrator near the steady-state can typically be very large. This scheme is in the following referred to as simulation approximated infinite horizon NMPC (SAIH-NMPC). Notice, that the results in (De Nicolao et al., 1996) are only given for discrete time systems, however they can be straightforwardly expanded to continuous time systems based on the results presented in (Mayne et al., 2000; Fontes, 2000b).

The approach proposed in (Chen and Allgöwer, 1996; Chen and Allgöwer, 1998b) uses an explicit upper bound of the infinite horizon cost inside of $\mathcal{E}$ that is obtained off-line. Typically the calculation of the terminal penalty E and the terminal region $\mathcal{E}$ are based on a linearization of the system and the use of a local linear control law. To obtain $\mathcal{E}$ and E a semi-infinite optimization problem must be solved off-line. The off-line calculation of $\mathcal{E}$ and E avoids the on-line integration of the system equations using the local control law for a long horizon. In the following we refer to this scheme as QIH-NMPC (quasi-infinite horizon NMPC).

3.1.2 Use of Suboptimal NMPC Strategies, Feasibility Implies Stability

To achieve stability it is often not necessary to find the global minima of the open-loop optimization problem. It is sufficient to achieve a decrease in the optimal cost at every time to guarantee stability (Chen and Allgöwer, 1998b; Scokaert et al., 1999; Jadbabaie et al., 2001; Fontes, 2000b; Findeisen, 1997; Findeisen and Rawlings, 1997). Thus, if one employs an optimization strategy that delivers feasible solutions at every sub-iteration while decreasing the cost, it is possible to stop the iterations whenever necessary and still guarantee stability.

3.2 Solution of the NMPC Optimal Control Problem

In principle, a wide variety of approaches for the solution of the open-loop optimal control problem (2.4) exists. In this chapter we mainly focus on the solution method proposed in (Diehl, Findeisen, Nagy, Bock, Schlöder and Allgöwer, 2002; Diehl, 2002; Findeisen, Diehl, Uslu, Schwarzkopf, Allgöwer, Bock, Schlöder and Gilles, 2002; Findeisen, Allgöwer, Diehl, Bock, Schlöder and Nagy, 2000), which has been implemented in a specially tailored version of the dynamic optimization package MUSCOD-II (Diehl, 2002; Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2002; Leineweber, 1998).

The NMPC optimal control problem (2.4) is introduced rather generally considering a terminal set $\mathcal{E}$ and input and state constraints sets. While this formulation is well suited for theoretical considerations, the (numerical) solution normally requires a concrete specification of the constraints in form of inequality or equality constraints. For the remainder of the chapter we assume that the appearing sets can be described by inequalities leading to the following formulation of the open-loop optimal control problem to solve in NMPC:

$$\min_{\bar{u}(\cdot)} J(\bar{x}(\cdot), \bar{u}(\cdot)) \tag{3.1a}$$

$$\text{subject to:} \quad \dot{\bar{x}} = f(\bar{x}, \bar{u}), \quad \bar{x}(t) = x(t) \tag{3.1b}$$

$$c(\bar{x}(\tau), \bar{u}(\tau)) \geq 0, \quad \tau \in [t, t + T_p] \tag{3.1c}$$

$$e(\bar{x}(t + T_p)) \geq 0. \tag{3.1d}$$

Here c defines the set of feasible states and inputs and e the feasible terminal region. Note that this reformulation does not change the overall setup and is only needed for describing the numerical solution strategies.

3.2.1 Solution by Direct Methods

There exist a variety of different approaches to solve the optimal control problem (3.1), see for example (Binder et al., 2001; Bryson and Ho, 1969; Vinter, 2000; Bertsekas, 2000). Typically so called direct solution methods (Binder et al., 2001; Biegler and Rawlings, 1991; Pytlak, 1999; Mayne, 1995; Diehl, Findeisen, Nagy, Bock, Schlöder and Allgöwer, 2002) are used, i.e. the original infinite dimensional problem is turned into a finite dimensional one discretizing the input (and possibly also the state).

Basically this is done by parameterizing the input (and possibly the states) by a finite number of parameters and to solve/approximate the differential equations during the optimization. In principle any parameterization of the input can be chosen, i.e. the parameterized input is given by

$$\bar{u}(\tau; q), \qquad \tau \in [t, t + T_p] \tag{3.2}$$

where the q is the vector of parameterization parameters. The parameterized $\bar{u}(\tau; q)$ might for example be given by a sum of basis functions such as a Fourier series or the input is parameterized as piecewise constant.

While the space of free parameters after the input parameterization is finite dimensional, the constraints on the inputs and states do lead to a semi-infinite optimization problem. Even so that the input constraints can often be rewritten as constraints on the input parameterization parameters leading to a finite number of input constraints, the state constraints are more difficult to capture. They are either enforced by adding an exact penalty term to the cost function or are approximately enforced at a finite number of time points over the prediction horizon. The resulting finite dimensional optimization problem takes the form:

$$\min_{q} J(\bar{x}(\cdot), \bar{u}(\cdot; q)) \tag{3.3}$$

subject to the state and input constraints and the system dynamics.

Mainly three strategies for the solution of the NMPC optimal control problem using mathematical programming can be distinguished (Pytlak, 1999; Binder et al., 2001; Biegler and Rawlings, 1991).

Sequential approach/feasible path approach:

In the sequential approach (de Oliveira and Biegler, 1994; Hicks and Ray, 1971; Kraft, 1985) the control is finitely parameterized in the form $\bar{u}(\tau; q)$ and the state trajectories are eliminated by numerically integrating the differential equation and cost. Only the control parameterization parameters remain as degree of freedom in a standard mathematical program given by (3.3). For each evaluation of the cost J in the solution of the mathematical program the differential equation and the cost function are numerically integrated using the current guess of the input parameterization parameters of the optimizer. Thus, the name sequential or "feasible path approach", since the optimization steps and the simulation are performed sequentially leading to a valid/feasible state trajectory. The sequential solution method is depicted in Figure 3.1.

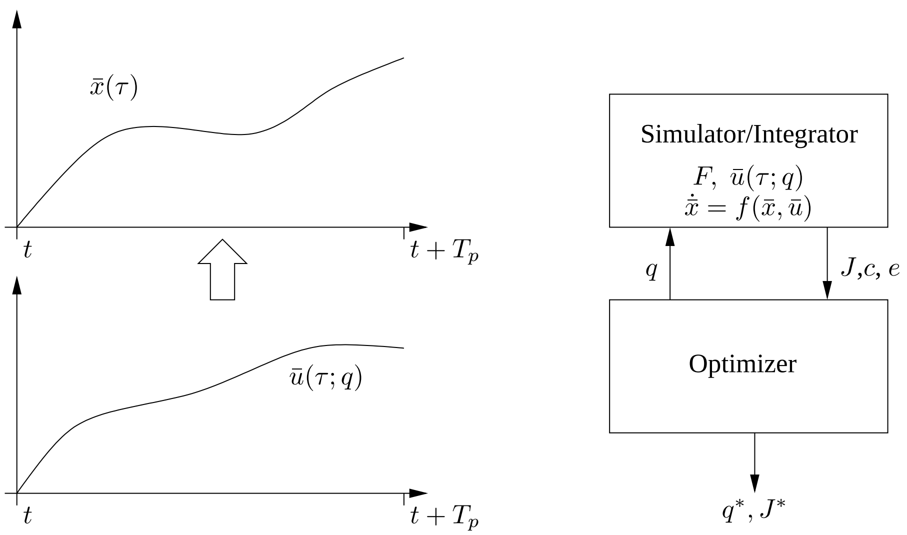


Figure 3.1: Sequential solution methods.

Simultaneous approach:

In the simultaneous approach the solution to the differential equation and the optimization is obtained simultaneously. For this purpose the differential equations are discretized and enter the optimization problem as additional constraints. Typical simultaneous approaches use collocation methods to parameterize/discretized the differential equations. In the collocation methods (Tsang et al., 1975;

Biegler, 2000; Cuthrell and Biegler, 1989) collocation is applied to the differential equations. The resulting nonlinear programming problem is very large but also rather sparse. This can be exploited to achieve an efficient solution.

Direct multiple shooting approach:
In the direct multiple shooting approach (Bock and Plitt, 1984; Tanartkit and Biegler, 1996; Leineweber, 1998; Bock, Diehl, Leineweber and Schlöder, 2000) the optimization horizon of interest is divided into a number of subintervals with local control parameterizations. The differential equations and cost on these intervals are integrated independently during each optimization iteration based on the current guess of the control. The continuity/consistency of the final state trajectory at the end of the

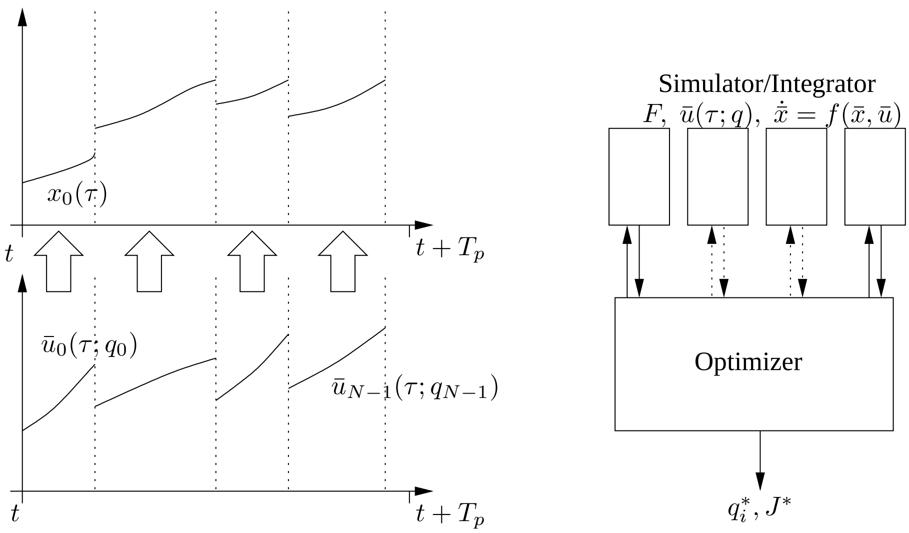


Figure 3.2: Simultaneous solution with multiple shooting.

optimization is enforced by adding consistency constraints to the nonlinear programming problem. The resulting nonlinear program takes a special sparse structure which can be utilized for an efficient solution.

Remark 3.1 *Besides the aforementioned direct solution approaches other approaches for the efficient solution of the open-loop optimal control problem exist, see for example (Binder et al., 2001; Bryson and Ho, 1969; Vinter, 2000; Bertsekas, 2000).*

We especially mention the class of approaches outlined in (van Nieuwstadt and Murray, 1998; Mahadevan and Doyle III, 2003; Petit et al., 2001) for differentially flat systems (Fliess et al., 1995; Fliess et al., 1999). These approaches utilize that for differentially flat systems a direct algebraic relation between the "output" and its derivatives and the input exists. This allows to reformulate the optimal control problem as a pure functional optimization. If the output is parameterized by suitably often differentiable basis functions, a static optimization problem results. However, the algebraic relation between the "output" and its derivatives and the input must be known explicitly. Furthermore, state and input constraints do complicate the considerations.

A closely related approach is based on ideas of input-output or input-to-state linearization. Basically the system is transformed to Byrnes Isidori normal form (Isidori, 1995) in which the appearing nonlinearity can be compensated by a suitable "compensation-term" (Nevistić and Morari, 1995; Primbs and Nevistić, 1997; Kurtz and Henson, 1997). The resulting system is linear in the transformed coordinates. Thus, a stabilizing linear MPC design, which can be solved computational efficiently, can be used. However, even linear state and input constraints in the original coordinates do result in nonlinear constraints that cannot be directly integrated in a linear predictive controller. Furthermore, the quadratic objective must be formulated in the transformed, often artificial coordinates.

The method exploited in the following is based on the direct multiple shooting approach.

3.3 Efficient Solution by Direct Multiple Shooting

In this section we review the direct multiple shooting approach and describe its application to the NMPC optimal control Problem 2. Furthermore, we describe important factors, which, if taken into account, can lead to a significant decrease of the required computation time. The resulting dynamic optimization scheme for NMPC is implemented in a special variant of the multiple shooting based dynamic optimization package MUSCOD-II (Leineweber, 1998). A detailed description of the implementation and numerics can be found in (Diehl, Findeisen, Nagy, Bock, Schlöder and Allgöwer, 2002; Diehl, 2002; Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2002). Note that while direct multiple shooting allows the consideration of index-one DAE systems (Bock and Plitt, 1984; Bauer et al., 1999), we do only consider ODEs here.

3.3.1 Basics of Direct Multiple Shooting

As in all direct solution methods, the input signal $u(\tau)$, $\tau \in [t, t+T_p]$ is approximated by a suitable finite parameterization. As outlined in the previous section in direct multiple shooting (also in most collocation methods) the input signal is defined on a disjoint multiple shooting grid on which locally supported control parameterizations are used. We assume that the multiple shooting grid is given by the partition π^o defined by

$$\tau_0 = t < \tau_1 < \tau_2 < \ldots < \tau_N = t + T_p. \tag{3.4}$$

Here and in the following the superscript o stands for optimization. Note that the partition $\pi^o_{[t,t+T_p]}$ of the optimization problem is in general independent of the partition π defining the recalculation instants of the NMPC controller and that the time $\delta^o_i = \tau_{i+1} - \tau_i$ between the shooting nodes τ_i does not have to be constant. To obtain a sufficiently good approximation of the infinite dimensional optimal control problem it is desirable to make $\bar{\pi}^o_{[t,t+T_p]} \ll \bar{\pi}$. Given this grid, the input on each of the multiple shooting intervals is given by the local input parameterization

$$\bar{u}_i(\tau; q_i), \quad \tau \in [\tau_i, \tau_{i+1}), \ \ i = 0, 1, \ldots N-1, \tag{3.5}$$

where $\bar{u}_i$ is a suitable basis-function parameterized in terms of the parameters q_i. The key requirement for the efficient solution is that the inputs $\bar{u}_i$ are only locally supported on each multiple shooting interval, i.e. the parameters q_i only influence the input on the interval $[\tau_i, \tau_{i+1})$. If continuity of the input between intervals is required, additional constraints can be introduced (Diehl, 2002; Leineweber, 1998). For simplicity of presentation we assume that the input is parameterized as piecewise constant, i.e.

$$\bar{u}_i(\tau; q_i) = q_i = \bar{u}_i. \tag{3.6}$$

The consideration of "independent" input parameterizations on the multiple shooting intervals is done to allow that the solutions of the system ODEs on these intervals can be considered as decoupled/independent from each other. For this purpose the initial conditions of the states at the beginning of each interval are introduced as additional degrees of freedom in the optimization problem, i.e. $N+1$ additional variables $\bar{s}_i \in \mathbb{R}^n$, $i = 0, \dots, N$ denoted as node values are introduced. Besides $\bar{s}_N$ all of these serve as initial values for the N decoupled initial value problems

$$\dot{\bar{x}}_i(\tau; \bar{s}_i, \bar{u}_i) = f(\bar{x}_i(\tau; \bar{s}_i, \bar{u}_i), \bar{u}_i), \quad \tau \in [\tau_i, \tau_{i+1}), \quad \text{with } \bar{x}_i(\tau_i; \bar{s}_i, \bar{u}_i) = \bar{s}_i \quad i = 0, \dots, N{-}1 \tag{3.7}$$

Given the values of $\bar{s}_i$ and $\bar{u}_i$ the solution of the N initial value problems define N trajectories $x_i(\tau; \bar{s}_i, \bar{u}_i)$, see Figure 3.3. The cost contribution of the multiple shooting interval i is given by

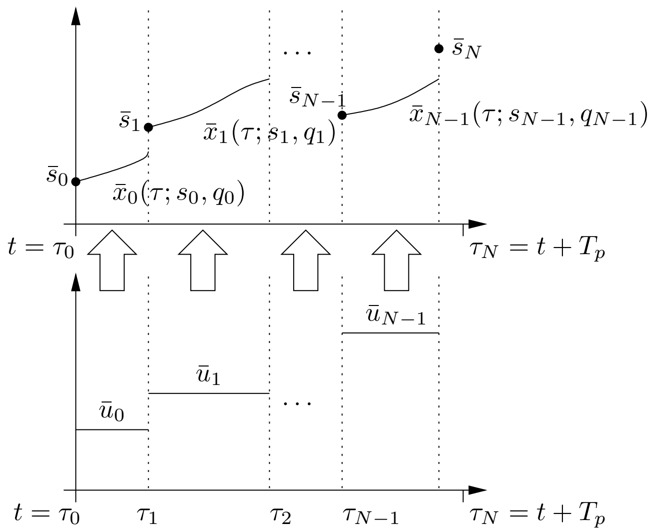


Figure 3.3: Multiple shooting considering N decoupled multiple shooting intervals and using a constant input parameterization given by the $\bar{u}_i$.

$$\Delta J_i(\bar{u}_i, \bar{s}_i, \tau_i, \tau_{i+1}) = \int_{\tau_i}^{\tau_{i+1}} F(\bar{x}_i(\tau; \bar{s}_i, \bar{u}_i), \bar{u}_i) d\tau. \tag{3.8}$$

The additional degrees of freedom $\bar{s}_i$ introduced do lead to a special structure in the equations appearing in the sub iteration of the resulting nonlinear program that can be utilized to achieve a fast solution. However, to obtain a consistent solution once the nonlinear program has converged, the consistency of the state trajectories must be guaranteed. This in done by introducing the following consistence equality constraints requiring that the final predicted state $\bar{x}_i(\tau_{i+1}; \bar{s}_i, \bar{u}_i)$ in each of the multiple shooting intervals matches the differential node value of the “next” interval $\bar{s}_{i+1}$

$$\bar{s}_{i+1} = \bar{x}_i(\tau_{i+1}; \bar{s}_i, \bar{u}_i), \quad i = 0, \ldots, N-1. \tag{3.9}$$

It is only required that these constraints are satisfied once the optimization algorithm used has converged. During the solution process they can be violated, allowing to use the additional degrees of freedom for a faster solution. Since the consistency condition (3.9) has to be satisfied once the algorithm has converged, the variables node values $\bar{s}_i$ introduced do not really represent additional degrees of freedom.

With respect to the state and input constraints often an approximation is used. The path constraints (3.1c) are approximated via $N+1$ inequality constraints at the multiple shooting nodes, i.e.

$$c(\bar{s}_i, \bar{u}_i) \geq 0, \quad i = 0, 1, \ldots, N. \tag{3.10}$$

While this does not guarantee satisfaction of the constraints in between the nodes, it is often satisfying in practice. Other sequential and simultaneous approaches use similar approximations.

The *finite dimensional* nonlinear program (NLP) resulting from the introduction of the node values $\bar{s}_i$ and the constraint approximation takes the following form:
NMPC Direct Multiple Shooting NLP

$$\min_{\bar{u}_i, \bar{s}_i} \left(\sum_{i=0}^{N-1} \Delta J_i(\bar{u}_i, \bar{s}_i, \tau_i, \tau_{i+1}) + E(\bar{s}_N) \right) \tag{3.11a}$$

subject to:

$$\bar{s}_{i+1} = \bar{x}_i(\tau_{i+1}; \bar{s}_i, \bar{u}_i), \quad i = 0, \ldots, N-1 \tag{3.11b}$$

$$\bar{s}_0 = x(t), \tag{3.11c}$$

$$c(\bar{s}_i, \bar{u}_i) \geq 0, \quad i = 0, 1, \ldots N \tag{3.11d}$$

$$e(s_N) \geq 0. \tag{3.11e}$$

This finite dimensional NLP has certain advantages with respect to the underlying structure that can be utilized for a fast numerical solution.

3.3.2 Solution and Properties of the Direct Multiple Shooting NLP

The multiple shooting NLP (3.1) is typically solved by a specially tailored sequentially quadratic programming (SQP) algorithm (Leineweber, 1998; Bock, Diehl, Leineweber and Schlöder, 2000; Bock and Plitt, 1984). Sequential quadratic programming is an iterative technique to find a point satisfying the so called Karush-Kuhn-Tucker (KKT) necessary conditions (see for example (Nocedal and Wright, 1999; Fletcher, 1987; Gill et al., 1981)) for a local optimum. In SQP a KKT point is found by iterating on a quadratic programming (QP) sub problem based on the Lagrangian of the system. For simplicity we rewrite NLP (3.1) in the following form:

$$\min_{w} F(w) \tag{3.12}$$

$$\text{subject to} \quad G(w){=}0 \tag{3.13}$$

$$H(w){\geq}0. \tag{3.14}$$

The vector w lumps the multiple shooting variables and controls:

$$w^T = \left[s_0^T, u_0^T, s_1^T, u_1^T, \ldots, s_N^T\right]. \tag{3.15}$$

The model dynamics/consistence constraints are contained in the equality constraint $G(w) = 0$, whereas $H(w)$ contains all (discretized) path and terminal constraints.

Initialized by an initial guess w_0, a SQP method for the solution of this NLP iterates by

$$w_{k+1} = w_k + \alpha_k \Delta w_k, \quad k = 0, 1, \ldots. \tag{3.16}$$

Here $\alpha_k \in \ (0, 1]$ is a relaxation factor, and the search direction Δw_k is given by the solution of the (QP) subproblem

$$\min_{\Delta w} \ \nabla F(w_k)^T \Delta w + \frac{1}{2} \Delta w^T A_k \Delta w \tag{3.17}$$

$$\text{subject to:} \quad G(w_k) + \nabla G(w_k)^T \Delta w{=}0 \tag{3.18}$$

$$H(w_k) + \nabla H(w_k)^T \Delta w{\geq}0 \tag{3.19}$$

The matrix A_k is a suitable approximation of the Hessian $\nabla_w^2 \mathcal{L}$ of the Lagrangian $\mathcal{L} = F(w) - \lambda_G^T G(w) - \lambda_H^T H(w)$, where λ_G and λ_H are the Lagrange multipliers.

Introducing the multiple shooting variables s_i and the control parameterization with local support leads to a special structure in the NLP problem and the resulting QP problems.

Specifically the (exact) Hessian of $\mathcal{L}$ has a sparse block diagonal structure. Similarly, the multiple shooting parameterization introduces a characteristic block sparse structure of the Jacobian matrices $\nabla G(w)^T$ and $\nabla H(w)^T$. We do not go into further details and refer to (Bock, Diehl, Leineweber and Schlöder, 2000; Diehl, Findeisen, Nagy, Bock, Schlöder and Allgöwer, 2002; Diehl et al., 2001). For performance and numerical stability it is of crucial importance that these structures of the NLP and QP are fully exploited. Crucial for an efficient solution are

- approximated Hessian updates should preserve the block diagonal structure of the exact Hessian.
- the QP solver used should exploit the block sparse structure.
- specialized robust and fast ODE (DAE) integrators should be used providing only the reduced gradients and Hessian blocks needed.

All these considerations have been taken into account in the dynamic optimization package MUSCOD-II (Leineweber, 1998) and in the specially adapted version for NMPC as described in detail in (Diehl, 1998; Diehl, Findeisen, Nagy, Bock, Schlöder and Allgöwer, 2002), which provides a flexible and efficient solution method for dynamic optimization problems.

3.3.3 Further Twists to Achieve Fast Solutions in the Case of NMPC

Between successive recalculation instants certain features can be utilized. For example, if a constant recalculation time $\delta_i = \delta^r$, which is equal to the length of the shooting intervals $\delta^o = \tau_{i+1} - \tau_i$, is used, than major parts of the input, state, Hessian and derivative information obtained at the recalculation time t_i can be reused at the next recalculation time t_{i+1} using a shifting strategy. Even if δ^r and δ^o are not equal and no shifting is performed, the values of the previous recalculation instant are a good initial guess at the next calculation.

More specifically, the following factors should be taken into account:
Use of Fast Integration Algorithms: As already mentioned, the use of special integrators is of crucial importance. The solution of the initial value problems and the corresponding derivatives are computed simultaneously by specially designed integrators which use the principle of internal numerical differentiation. In particular, the integrator DAESOL (see (Bauer et al., 1997; Bauer, 2000)), which is based on the backward-differentiation-formula (BDF), is used in the special NMPC MUSCOD-II implementation.
Initial Value Embedding Strategy (Bock, Diehl, Schlöder, Allgöwer, Findeisen and Nagy, 2000): Optimization problems at subsequent recalculation instants differ only by different initial values that are imposed via the initial value constraint $\bar{s}_0 = x(t_i)$. Accepting an initial violation of this constraint, the solution trajectory of the previous optimization problem can be used as an initial guess for the current problem. Furthermore, all problem functions, derivatives as well as an approximation of the Hessian matrix are already available for this trajectory and can be used in the new problem, so that the first QP solution can be performed without any additional ODE solution. This approach differs from a conventional warm start techniques for NMPC (Biegler and Rawlings, 1991; Liebman et al., 1992), which typically initialize the NLP variables by integrating the ODE with the old (or shifted) input and the current $x(t)$.
Efficient Treatment of Least Squares Cost Functions: An efficient approach to obtain a cheap Hessian approximation – the constrained Gauss-Newton method – is recommended in the special case of a least squares type cost function. In NMPC, the involved least squares terms arise in integral form $\int_{t_j}^{t_{j+1}} \|l(x,u)\|_2^2 \, dt$. Specially adapted integrators that are able to compute a numerical approximation of the Gauss-Newton Hessian for this type of least squares term have been developed (Diehl et al., 2001).

The consideration of these factors does improve robustness and speed of the optimization algorithm significantly and have been implemented in the special NMPC version of MUSCOD-II, see (Diehl, Findeisen, Nagy, Bock, Schlöder and Allgöwer, 2002; Diehl et al., 2001).

3.4 Control of a High-Purity Distillation Column

In the following, we apply the derived solution method for the NMPC optimal control problem in simulations and experiments considering the control of a high purity binary distillation column.

The column has 40 trays and we consider the separation of Methanol and n-Propanol. The binary mixture is fed into the column (compare Figure 3.4) with the flow rate F and the molar composition x_F. The products are removed at the top and bottom of the column with the concentrations x_D and

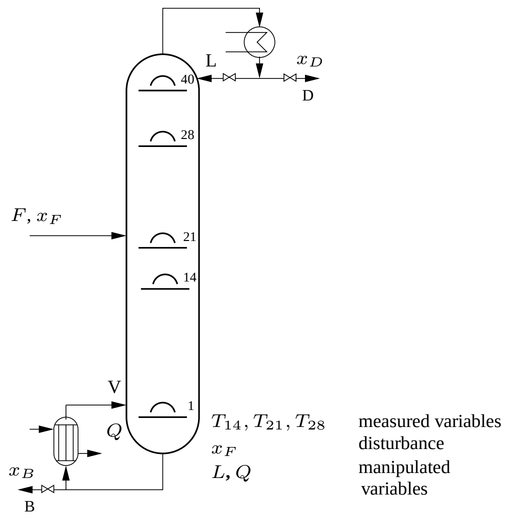


Figure 3.4: High purity Distillation column.

x_B. The manipulated variables are the reflux flow L and the vapor flow V (simulation) or the heat input Q to the column (experiments). If not otherwise mentioned, we assume all states as directly available for state-feedback. The control problem is to maintain the specifications on the product concentrations x_B and x_D despite the occurrence of disturbances.

Different models for the distillation column are available. Modeling of the distillation column under the assumption of constant relative volatility, constant molar overflow, no pressure losses, no energy balances and hydrodynamics leads to a 42^{nd} order ODE model. The states are the concentrations on the trays, in the reboiler and in the condenser. Based on this model a reduced 5^{th} order ODE model utilizing the so called wave propagation phenomena is available (Rehm and Allgöwer, 1996; Findeisen and Allgöwer, 2000a), that has as states the concentrations in the reboiler, condenser and feed trays as well as the wave positions for the stripping and rectifying sections, respectively. Furthermore a 164^{th} order model with 42 differential states (concentrations on the trays) and 122 algebraic states (liquid flows, vapor flows and temperatures on each tray) is available. Detailed descriptions of these models can be found in (Nagy et al., 2002; Nagy, Findeisen, Diehl, Allgöwer, Bock, Agachi and Schlöder, 2000).

3.4.1 Simulation Results

For all simulations the plant is given by the 164^{th} order model and the recalculation time is fixed to $\delta^r = 30s$. All calculations are carried out on a Digital Alpha XP 1000 Workstation using the described specially tailored version of MUSCOD-II. If not otherwise stated the time span δ^o between two multiple shooting nodes is constant and the same as the recalculation time δ^r of the NMPC controller.

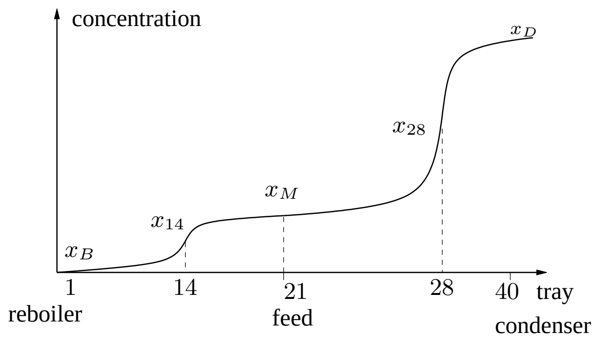


Figure 3.5: Concentration wave profi le.

Basic controller setup:
The control problem is to maintain the specifications on the product concentrations x_B and x_D despite the occurrence of disturbances. In the cost function F the quadratic deviation of these variables and the inputs from their steady-state values are weighted. As usual in distillation control, x_B and x_D are not controlled directly. Instead an inferential control scheme, which controls the deviation of the concentrations on tray 14 and 28 from the set-points is used. Since for the standard set-point conditions the turning point positions of the waves approximately correspond to these trays, one can expect good control performance with respect to x_B and x_D, compare Figure 3.5. Even small changes in the inflow or feed conditions lead to significant changes in the wave positions and thus of the concentrations on trays 14 and 28. However, the changes in the product concentrations x_B and x_D are comparable small. Thus, if the concentrations at the turning points are controlled well, one can expect that the product concentrations are satisfying.

Due to these considerations only the concentration deviations from the set-point on trays 14 and 28 are penalized in the stage cost-function F:

$$F(x,u) = \left\| \begin{bmatrix} x_{14} - x_{14s} \\ x_{28} - x_{28s} \end{bmatrix} \right\|^2_{Q_w} + \left\| \begin{bmatrix} L - L_s \\ Q - Q_s \end{bmatrix} \right\|^2_R . \tag{3.20}$$

To avoid offset between the different controllers due to model plant mismatch the steady-states used in the controller have been adjusted accordingly to guarantee offset free control for the nominal steady-state. Furthermore, for comparability it is assumed that the disturbances in the feed concentration can be measured and thus are known by the controller if not otherwise stated.

Comparison of different NMPC schemes:
In this section we compare infinite horizon NMPC with QIH-NMPC and SAIH-NMPC as introduced in Section 3.1. All three controllers use the 164^{th} order model for prediction (no model plant mismatch). For the calculation of the terminal region and terminal penalty term for the QIH-NMPC approach and the SAIH-NMPC approach, we use a LQR controller. This controller is derived on the basis of the linearization of the system around the considered steady-state. Since this controller has a rather large region of attraction, no terminal region is considered in the SAIH-NMPC approach.

The terminal region and the quadratic upper bound for the QIH-NMPC approach is found by direct optimization (Findeisen and Allgöwer, 2000a; Findeisen and Allgöwer, 2000a). The infinite horizon NMPC scheme was approximated by an NMPC scheme with 40 control intervals plus a prediction interval of $10000s$ at the end, where the input was fixed to the steady-state value. Simulation experiments showed that the performance does not change much if more than 40 intervals are used. For the SAIH-NMPC approach 5 control intervals were used. The end penalty term E was calculated by simulating the system with the local linear controller for another $10000s$ at the end of the control horizon. In the QIH-NMPC scheme also 5 control intervals were used. To allow the upper bounding of the infinite horizon cost by the (quadratic) penalty term, the final predicted state was constrained to lie in the quadratic terminal region.

We exemplary show the simulation results for a rather drastic scenario, see Figure 3.6. At $t = 270$s the reflux is decreased from 4l/h to 0.5l/h (reflux breakdown), while the vapor flow V is kept constant, i.e. the manipulated variables are fixed to these values. After 180s the controller is switched on again. This leads to a rather large offset from the steady-state. All three controllers nicely return

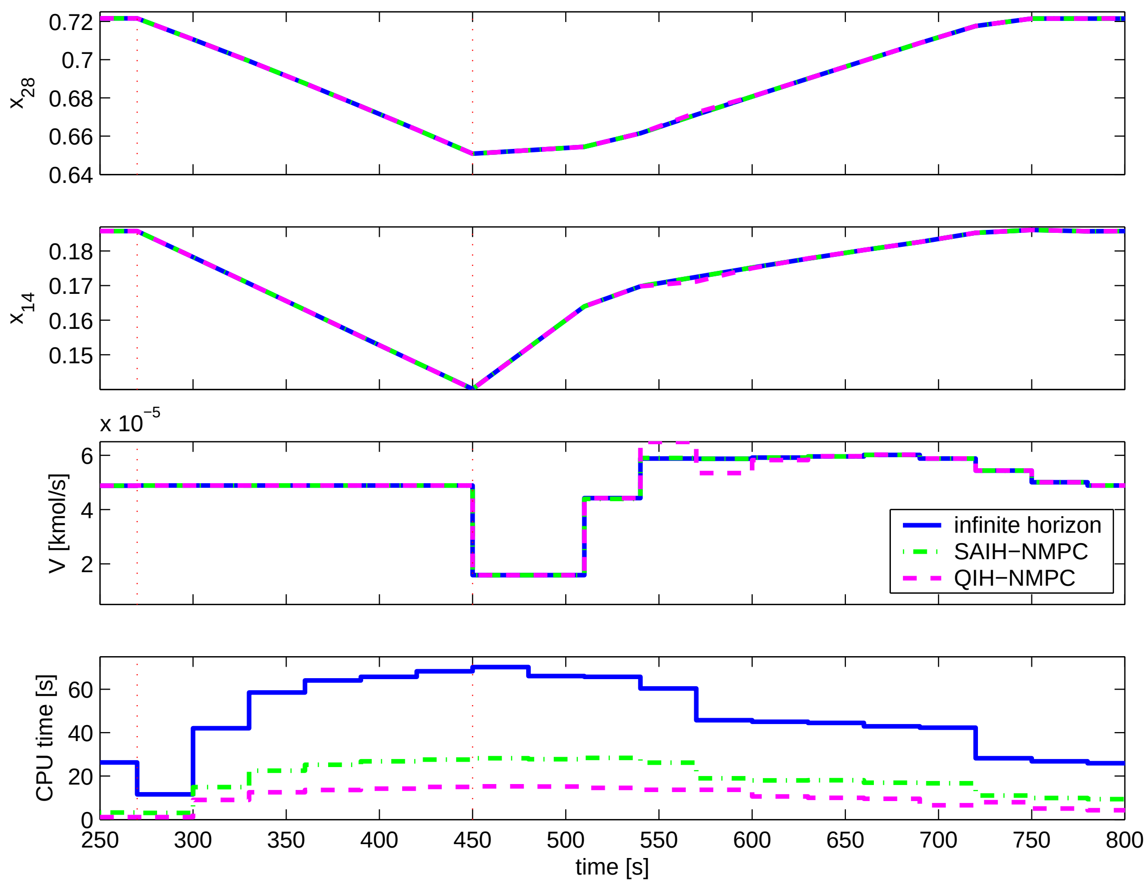


Figure 3.6: Performance and computational demand of different NMPC schemes.

the system to the steady-state. Not surprisingly the best performance is achieved by the "infinite" horizon controller. However, this controller cannot be implemented in real-time, since the solution of one open-loop optimal control problem in general requires more than the 30 seconds, compare

Figure 3.6. In comparison SAIH-NMPC approach as well QIH-NMPC controller are able to meet the real-time requirement. As expected, the QIH-NMPC approach does lead to a small degradation in the performance. However, the safety margin to the maximum allowed CPU time is significantly higher as for the SAIH-NMPC approach.

Influence of the model size on performance and computational demand:
In this part the influence of different model sizes and prediction horizon length on the necessary computation time for the QIH-NMPC scheme are examined. The controller setup is the same as in the previous section. Table 3.1 shows the average and maximum CPU time for a disturbance scenario in the feed concentration x_F. One can see that the QIH-NMPC scheme using MUSCOD is feasible

Table 3.1: Comparison of the necessary CPU time.

model	N=5 (150s)		N=10 (300s)		N=20 (600s)	
size	max	avrg	max	avrg	max	avrg
5	0.4	0.1	1.9	0.3	5.8	0.6
42	0.9	0.4	2.1	0.8	6.8	2.0
164	18.8	1.9	36.6	4.5	47.5	5.2

for the 5^{th} and 42^{nd} order models even for a prediction horizon of 600s ($N = 20$). Even in the case of the 164^{th} order model the predictive controller is real-time applicable if a prediction horizon of $N = 5$ is used. The CPU time only grows "linearly" with the horizon length. For comparison, the solution time for a zero-terminal constraint NMPC scheme requires a minimum horizon length of $N = 20$ to be feasible and the solution time for the 5^{th} order model increase to 6.8 seconds. Thus, the use of suitable NMPC strategies that require a reduced computational load has a significant influence on the necessary solution time.

Computational delay and state estimation:
In this part we briefly outline the influence of a state observer and the computational delay due to the solution time of the optimal control problem on the closed-loop. This is mainly done in preparation for the experimental results presented in Section 3.4.2 and for the considerations in Section 4.5.

For this purpose we consider again the QIH-NMPC as described in Section 3.1. The simulation setup is similar to the experimental setup presented in the next section. So far we assumed that all states can be accessed directly by measurements. In reality this is however not possible. In the case of the pilot scale distillation column we assume that only the feed tray temperature and the temperature on the 14^{th} and 28^{th} tray are directly measurable. The remaining states are recovered by an Extended Kalman Filter (EKF). Also the unknown "disturbances" x_F and F are estimated by the EKF by augmenting the system model by two additional integrators.

Based on the system parameters estimated by the EKF in a first step, the system state at the next recalculation instant is predicted. The open loop optimal control problem is then solved for the

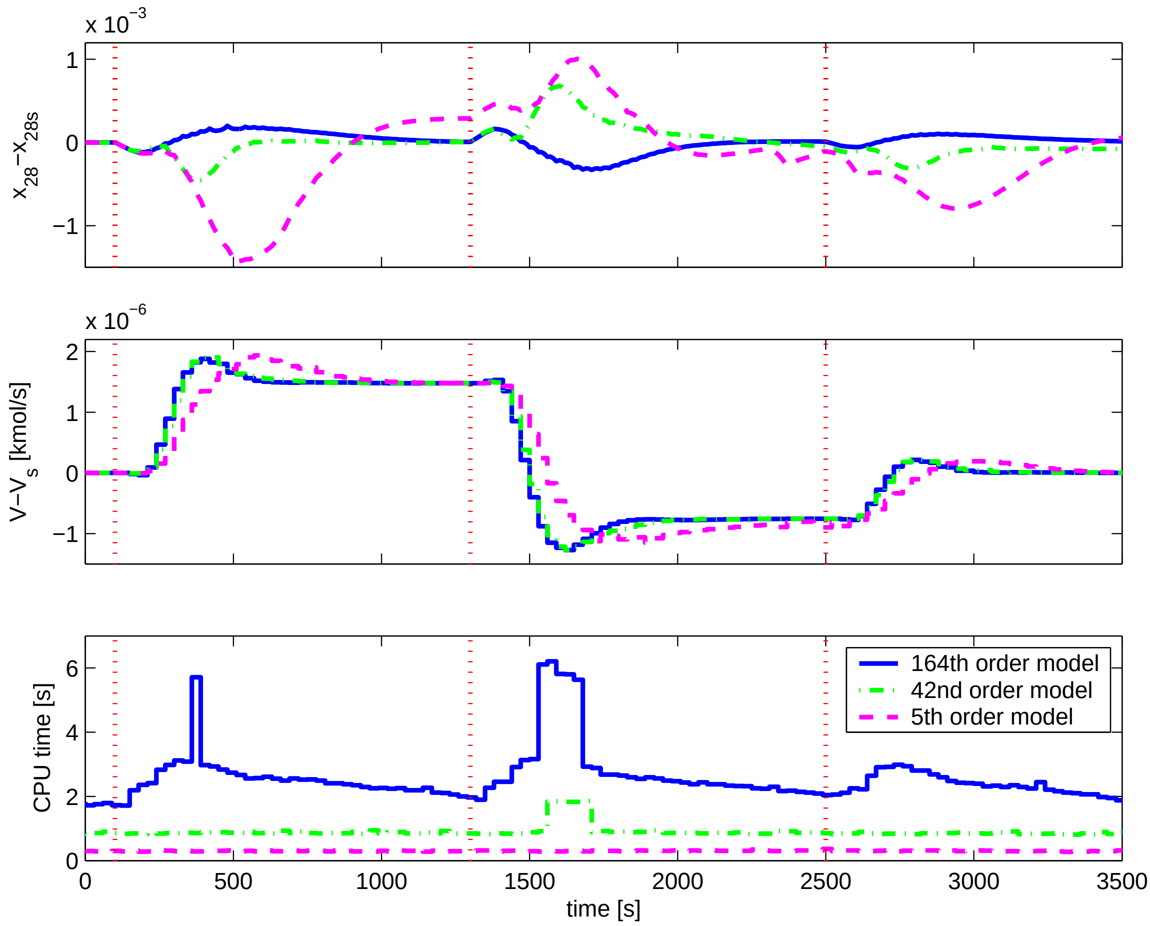


Figure 3.7: Behavior of the closed-loop including an EKF for state estimation.

predicted state. The resulting first input is implemented at the next control instant and the procedure is repeated. This is necessary, since the solution of the dynamic optimization problem cannot be obtained instantaneously. However, as shown in Chapter 4, if the computational delay is considered in this way, the stability properties remain the same as in the nominal case.

Figure 3.7 shows the behavior of the closed-loop considering three different model sizes with respect to feed concentration disturbances. The concentrations are nicely kept in a narrow band around the desired steady-state even so that the EKF is used for the state recovery. One can see that the solution is easily possible even for the 164^{th} order model. The reduced maximum computational time in comparison to the state-feedback case examined in the section before is mainly due to the "smoothing" effect of the EKF. Since the disturbance and the system state are estimated by the observer and thus do not change instantaneously, the optimization problems also change only little from recalculation time to recalculation time. Thus, often one SQP iteration is sufficient to obtain a new optimal solution.

This also stipulates the use of the so called *real-time iteration* scheme (Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2002), which only performs one SQP iteration per recalculation instant, see Section 3.5.

3.4.2 Experimental Verifications

In this section we present experimental results verifying the obtained simulation results.

The experimental implementation was carried out on a pilot plant distillation column situated at the Institut für Systemdynamik und Regelungstechnik of the University of Stuttgart. It has a diameter of 0.1m and a total height of 7 m and temperature measurements on all trays and in the reboiler as well as measurements of the flows are available. The overhead vapor is totally condensed in a water cooled condenser, which is open to the atmosphere. The reboiler is heated electrically. Thus, in difference to the simulation results now the volumetric liquid reflux flow L_{vol} of the condenser (which can be measured) and the heat input Q to the boiler are the manipulated variables.

The column is coupled to a process control system. All computations are performed on a standard PC running Linux (AMD Athlon, 1100Mhz). The data communication with the process control system is performed via ftp, allowing a reliable input/output operation all 10s.

First experiments showed that in order to achieve good control performance, the hydrodynamics in the column should not be neglected. Thus, based on the 164^{th} order model a 204^{th} order model (122 algebraic states and 82 differential states) including hydrodynamics is derived (Diehl et al., 2001; Diehl, 2002). Some of the model parameters where estimated based on experimental data using a special off-line version of MUSCOD-II.

The stage cost function F now weighs the temperatures instead of the concentrations, which cannot be directly measured. The states are estimated from the temperature measurements on the 14^{th}, 28^{th} and 21^{th} (feed) tray using an extended Kalman filter. Furthermore, the EKF is fed with the measurements of the volumetric feed flow F_{vol} and the manipulated variables are L_{vol} and Q. The overall NMPC based controller setup is shown in Figure 3.8. This scheme was implemented and compared to a

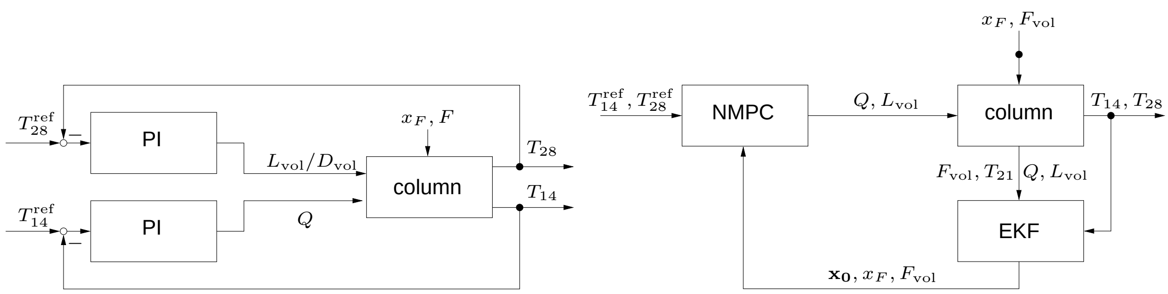


Figure 3.8: Setup PI controller (left) and NMPC controller (right) for the experimental validation.

conventional PI control scheme. The PI control scheme usually employed to control the column consists of two decoupled SISO PI loops. One uses the heat input Q to control the temperature on the 14^{th} tray, the other uses the reflux L to control the temperature T_{28}. The setup of the PI loop is also shown in Figure 3.8. Figure 3.9 exemplary shows the temperature on the 28^{th} tray as well as the heat input Q to the column for both controllers. Starting from a steady-state, the feed flow F is increased at $t = 1000$s by 20 percent. The NMPC controller is able to complete the transition into the new steady-state in approximately 1000s with a maximum deviation of T_{28} of 0.3^oC. Even though

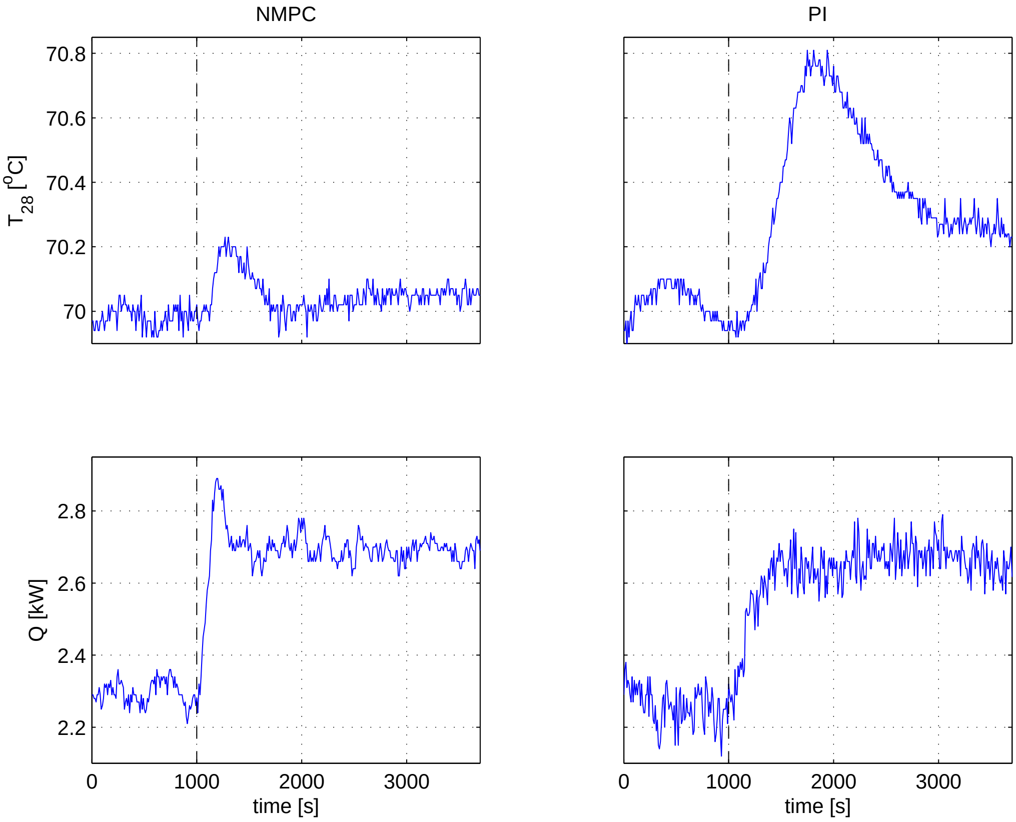


Figure 3.9: Experimental results NMPC and PI controller.

no extensive tuning of the NMPC controller was performed, the temperature only shows a maximum deviation of 0.8^oC and completes the transition to the new steady-state with an inevitable offset of 0.25^oC in T_{28} after 1500s. A detailed discussion of the experimental results obtained can be found in (Diehl et al., 2001; Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2003).

The presented results underpin that an application of NMPC schemes with reduced computational demand is possible even nowadays, if specially tailored optimization schemes are used. Furthermore, the closed-loop shows good performance without the need to use a reduced order model or much tuning.

3.5 Efficient Solution via the Real-time Iteration Scheme

So far we assumed that the SQP algorithm used for the solution of the NMPC optimal control problem is iterated until convergence. As already briefly mentioned, often optimization problems from one recalculation instant to the next do only differ minimally. Thus, as proposed in (Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2002; Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2003; Diehl, Findeisen, Nagy, Bock, Schlöder and Allgöwer, 2002) it might be sufficient to actually stop iterating the SQP before convergence. Basically it is proposed to:

- Firstly perform only one SQP iteration per recalculation instant, allowing to decrease the recalculation time δ^r which is for the full iteration scheme mainly limited by the time required to solve the NLP. As shown in the simulation results in Section 3.4.1 often even one iteration is sufficient to nearly converge. Also the approach is satisfactory if one already starts sufficiently close to a valid solution and maintains the convergence properties of direct multiple shooting.

- Secondly, to decrease the computational delay between the state measurement and the availability of a new valid input, to divide the calculation in a preparation phase and a fast feedback phase. In the preparation phase all the time consuming preparations for one SQP step such as integration of the differential equations (for example to obtaining $G(w)$ in the SQP and its derivatives) are performed. In the fast feedback phase only minor calculations are performed leading to the new input based on the current measurement.

Further details can be found in (Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2003). In principle, the error introduced by such a real-time iteration scheme can be considered as an external disturbance. In Chapter 5 it is shown that the closed-loop using sampled-data NMPC is robustly stable if the disturbances (e.g. difference to the optimal solution) is sufficiently small. The stability and convergence of the real-time iteration scheme is analyzed in detail in (Diehl et al., 2004; Diehl, Findeisen, Allgöwer, Schlöder and Bock, 2003). There it is shown that closed-loop stability under the real-time iteration scheme for discrete time systems is possible if the initial solution is sufficiently close to the optimal solution manifold.

Figure 3.10 underpins that the application of the real-time iteration scheme to the distillation control problem does lead to significant reduction of the required CPU time and good performance. Figure 3.10 shows the necessary CPU time and performance for a full iteration NMPC scheme and a real-time iteration strategy. For comparison a similar situation to the one shown in Figure 3.9 is considered.

3.6 Summary

In this chapter we examined the question if the optimal control problem appearing in NMPC can be solved efficiently, allowing for a real-time application of NMPC to realistically sized problems. Specifically we outlined a tailored efficient mathematical programming based solution strategy using direct multiple shooting as derived in the context of an NMPC real-time feasibility study (Nagy, Findeisen, Diehl, Allgöwer, Bock, Agachi, Schlöder and Leineweber, 2000; Diehl, 1998; Diehl, Findeisen, Nagy, Bock, Schlöder and Allgöwer, 2002; Diehl et al., 2001; Findeisen, Allgöwer, Diehl, Bock, Schlöder and Nagy, 2000; Findeisen, Nagy, Diehl, Allgöwer, Bock and Schlöder, 2001; Findeisen, Diehl, Bürner, Allgöwer, Bock and Schlöder, 2002; Findeisen, Diehl, Uslu, Schwarzkopf, Allgöwer, Bock, Schlöder and Gilles, 2002). To show the computational efficiency of this scheme we presented simulation and experimental results for the control of a high-purity distillation column for the separation of a binary mixture. The main result of this section is that from a computational point of view NMPC can even be nowadays applied to practically relevant processes. However, to allow for real-time feasibility, one should use NMPC schemes that facilitate an efficient solution and utilize specially tailored dynamic optimization schemes.

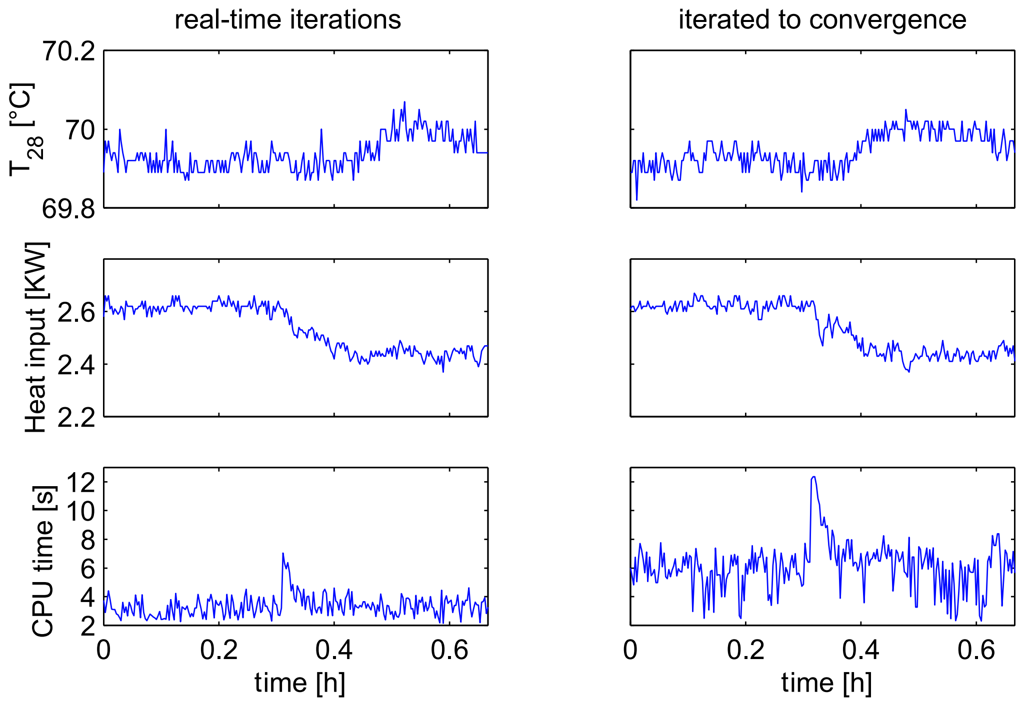


Figure 3.10: Performance and CPU time comparison of real-time iteration scheme to full iteration scheme.

Note that the derived efficient solution approaches can be expanded to the optimization based state estimation problem via moving horizon state estimation, as outlined in (Bürner, 2002; Findeisen, Diehl, Bürner, Allgöwer, Bock and Schlöder, 2002).

While the presented results underpin the real-time applicability of NMPC from a computational viewpoint, a series of practical and theoretical questions remain open. Specifically it is not clear, under which conditions nominal stability and performance results for the closed-loop do hold in practice. Some of the issues are:

- *Computational and measurement delays:* the numerical solution of the open-loop optimal control problem introduces a delay between the measured state and the implemented input, which is often not taken into account. Furthermore, state and output measurement delays might be present that should be taken into account or compensated.
- *Model plant mismatch:* the real plant often differs from the nominal plant model used for predictions.
- *External disturbances:* external disturbances which are not taken into account.
- *Numerical errors:* the numerical solution of the optimal control problem introduces approximation errors in comparison to the nominal optimal input.
- *Output-feedback:* predictive control based on state space models is inherently a state-feedback scheme. The system state is necessary for the prediction.

In the following chapters we investigate these issues in a more general framework, considering the control of continuous time systems via sample-data open-loop feedback.

Chapter 4

Stability of Sampled-data Open-loop State-feedback

The results of the previous chapter underline that computationally the practical application of NMPC is possible even nowadays. However, there are a series of open theoretical question with respect to sampled-data open-loop NMPC. The remaining chapters provide answers to some of these questions. Specifically we examine issues related to the stability of the closed-loop in the case of numerical approximation errors in the solution of the optimal control problem, external disturbances, uncertain parameters, model-plant mismatch, inevitable computational delays, measurement delays, and stability conditions for the sampled-data open-loop output-feedback problem.

Most of the derived results are not limited to sampled-data NMPC. For this reason we consider in the following a more general setup, the control of nonlinear systems using sampled-data open-loop feedbacks. NMPC can be seen as one specific representative of this class of controllers.

In this chapter we specifically focus on the derivation of stability conditions for sampled-data open-loop feedback. After a short introduction of the considered sampled-data open-loop setup, we derive in Section 4.3 stability conditions that guarantee stability of the closed-loop. Notably, the derived results allow for varying recalculation intervals and the consideration of constraints on inputs and states. Furthermore, the results are not limited to controls that are continuous in the state. This allows considering discontinuous feedbacks, as might for example be necessary for the control of nonholonomic systems (Brockett, 1983; Fontes, 2003; Clark, 2001; De Luca and Giuseppe, 1995; Astolfi, 1996; Ryan, 1994). Section 4.4 presents two control approaches for which the derived stability conditions are directly applicable. Section 4.4.1 shows that asymptotically stabilizing locally Lipschitz continuous instantaneous feedbacks can be adapted to sampled-data open-loop feedback by feedforward simulation. This allows applying instantaneous feedbacks even in the case of state information available only at the recalculation instants. Section 4.4.2 outlines the application of the derived stability result to sampled-data NMPC.

Besides the question of nominal stability we furthermore consider the practically important question of measurement and computational delays. As shown in Section 4.5, sampled-data open-loop feedback allows a rather simple consideration of computational and measurement delays without loss of stability.

The obtained results are exemplified in Section 4.4.3 and 4.5.3 considering the control of a continuous stirred tank reactor.

The derived nominal stability results lay the basis for the robustness and output-feedback considerations in Chapter 5 and Chapter 6. The key observation utilized is that the nominal decrease in the "sampled-data Lyapunov function" provides, under certain continuity assumptions, robustness with respect to small external disturbances and excitations.

4.1 Sampled-data Feedback and Sampled-data Open-loop Feedback

Classical sampled-data control for continuous time systems refers to the control of a continuous time plant using a discrete time feedback (Chen and Francis, 1995; Aström and Wittenmark, 1997; Franklin et al., 1998) or vice versa. This problem is motivated by the fact that most controllers are implemented using microprocessors. Typically, the interconnection between the discrete and continuous time is achieved using suitable A/D and D/A converters (often referred to as sampler and zero-order holds), see Figure 4.1.

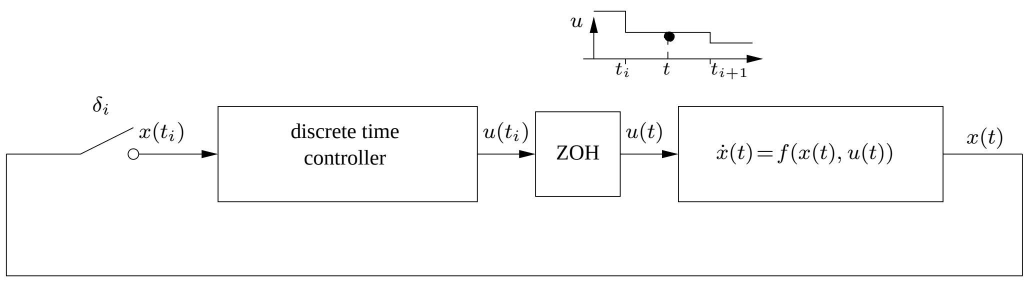


Figure 4.1: Sampled-data feedback.

Sampled-data control has received significant interest in recent years, see for example (Nešić and Teel, 2001; Nešić and Laila, 2002; Hou et al., 1997; Chen and Francis, 1995) and references therein.

The main issue in sampled-data control for nonlinear systems is that for a continuous time nonlinear system it is in general not possible to derive an exact discrete time model. Thus, for the design of the controller one either has to use an approximated model to design the controller in discrete time, or, after designing a continuous time controller, one implements an approximated version of this controller in discrete time (Nešić and Teel, 2001; Chen and Francis, 1995). The elementary question for both design methods is, if the properties achieved in the design in one of the domains are preserved in the other domain. One specific question is, if the stability of the closed-loop is retained even if the controller is designed in continuous time, but implemented approximately in discrete time. With respect to these questions a series of results have been obtained, see e.g. (Nešić and Teel, 2001; Nešić and Laila, 2002; Nešić et al., 1999).

In comparison to the classical sampled-data control for continuous time systems we consider a slightly modified problem. Specifically, we do not consider that the applied input is sampled/kept constant in between the recalculation instants, compare Figure 4.2. Furthermore, we do not assume that the

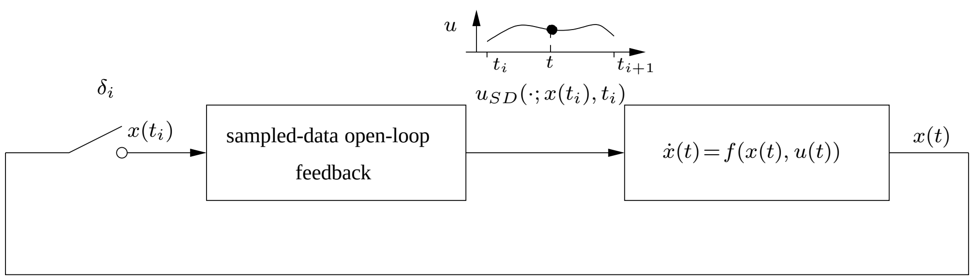


Figure 4.2: Sampled-data open-loop feedback.

recalculation times are constant.

There are several reasons for not considering to sample-and-hold the input in between the recalculation times. Firstly microprocessors and A/D and D/A converters are becoming faster and faster. Frequently, the speed of the A/D and D/A converters/microprocessors are not the limiting factors for practical implementations anymore, at least for control problems typically encountered in the process industry[1]. By now there even exist process control systems allowing the direct consideration and use of continuous time controller representations. The corresponding differential equations are then numerically integrated on-line.

Rather than the speed of the A/D and D/A converters, typically slow state "measurements" are key limiting factors. Slow state measurements might for example be due to slow sensors such as concentration measurements, or due to the required extraction of the state information from secondary measurements involving for example computationally intense image processing. Furthermore, the recalculation time might be, as for example in the case of NMPC, dictated by the time required to solve a computationally expensive optimal control problem. Typically, the sampling time (in the following denoted by δ^S) of the process control system, at which the A/D and D/A converters operate, is in the order of milli- or even micro-seconds, whereas the recalculation time and availability of sensor measurements might be in the order of seconds. If in this case the input is kept constant in between recalculation instants, the achievable performance can degrade significantly. One possibility to overcome this problem is to open-loop apply an input signal obtained at the recalculation time t_i. Even so the D/A converters/sample-and-hold elements will lead to an approximation error of the open-loop input, these effects can often be neglected, compare Figure 4.3. The resulting approximation error can rather be considered as a (small) disturbance, which sampled-data open-loop feedbacks are, under certain conditions, able to reject, see Chapter 5.

A closely related aspect of considering continuous inputs instead of "constant" ones is that constant inputs with fixed recalculation/sampling time do limit the achievable performance in the sense that

[1]The situation is different for control problems typically encountered in the aerospace or automobile industry. Such processes often require a feedback response in the order of milliseconds or even microseconds.

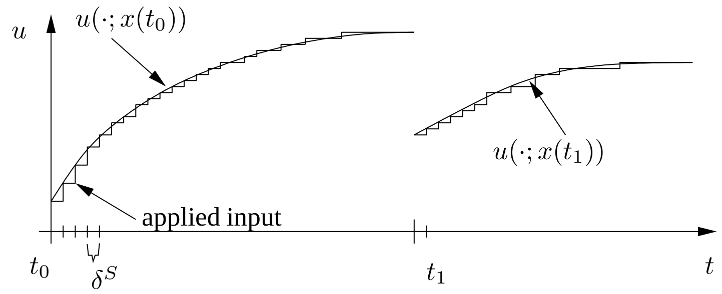


Figure 4.3: Recalculation time, sampling time and sample-and-hold.

asymptotic convergence to the origin/reference trajectory can only be achieved by decreasing the recalculation time to zero (Clarke et al., 2000; Fontes, 2003).

One question occurring is, whether there are any controller designs available that can provide for a single state measurement an input signal rather then a “fixed” input value? Luckily, by now a whole series of controller design exist that can provide open-loop input trajectories for one single state measurement. A classical example is optimal control. Further examples are sampled-data open-loop NMPC (Fontes, 2000b; Findeisen et al., 2003e) or open-loop input generators as outlined in (Alamir and Bonard, 1999; Marchand and Alamir, 1998), which might for example be based on differential flatness or other structural considerations. Furthermore, as shown in Section 4.4.1, any stabilizing instantaneous feedback can be used to obtain suitable open-loop input trajectories by feedforward simulation.

Summarizing, in comparison to classical sampled-data control, where typically the input is kept constant in between recalculation instants, we consider the case that the applied input in between recalculation instants is given by a “continuous” signal. The unavoidable approximation effects of sample-and-hold elements, which are present in any digital implementation, are assumed to be small and can be considered as small disturbances acting on the closed-loop system, compare Chapter 5.

4.2 Basic Setup

In this chapter we consider time-invariant nonlinear systems given by

$$\dot{x}(t) = f(x(t), u(t)) \qquad t \geq 0, \;\; x(0) = x_0 \in \mathcal{X}_0, \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ denotes the system state. The input is denoted by $u(t) \in \mathcal{U}$ a.e., and $\mathcal{X}_0$ denotes the set of considered initial conditions. Here the set $\mathcal{X} \supseteq \mathcal{X}_0$ denotes the set of admissible states and $\mathcal{U} \subseteq \mathbb{R}^m$ denotes the set of admissible inputs. The vector field $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is assumed to satisfy

Assumption 4.1
$f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ *is continuous in its arguments and locally Lipschitz continuous in* x.

Our objective is to derive stability conditions for system (4.1) under sampled-data open-loop feedback of the form:

$$u(t) = u_{SD}(t; x(t_i), t_i). \tag{4.2}$$

Here u_{SD} denotes the open-loop input trajectory of the sampled-data feedback controller as defined later. It is based on the state information $x(t_i)$ at the recalculation instant t_i, compare Figure 4.2.

Remark 4.1 *We only consider "static" feedbacks, i.e. we assume that the input generator does not have an internal state, i.e. it only depends on the state $x(t_i)$ at the recalculation time. This is mainly done for simplicity and based on the fact that predictive controllers as well as other existing suitable open-loop input generators (Alamir and Bonard, 1999; Marchand and Alamir, 1998) do not possesses an internal controller dynamics. However, it is possible to expand the results to dynamical controllers, see Section 4.3.1.*

We refer to an admissible input generator as

Definition 4.1 *(***Admissible input generator***)*
An input generator is called admissible with respect to the sets $\mathcal{X}_0 \subseteq \mathcal{X} \subseteq \mathbb{R}^n$, $\mathcal{U} \subseteq \mathbb{R}^m$, and a partition π, if for any $x \in \mathcal{X}_0$ and any $t_i \in \pi$

1. $u_{SD}(\cdot; x, t_i) \in \mathcal{L}^\infty([t_i, t_{i+1}], \mathcal{U})$
2. *the solution $x(\cdot; x(t_i), u_{SD}(\cdot; x(t_i), t_i))$ of* (4.1) *under the input u_{SD} starting from $x(t_i)$ is absolutely continuous on $[t_i, t_{i+1})$ with*
   (a) $x(\tau; x(t_i), u_{SD}(\cdot; x(t_i), t_i)) \in \mathcal{X}\ \forall \tau \in [t_i, t_{i+1})$
   (b) $x(t_{i+1}; x(t_i), u_{SD}(\cdot; x(t_i), t_i)) \in \mathcal{X}_0$.

Here $\mathcal{L}^\infty([a, b], \mathcal{U})$ denotes Lebesgue measurable and essentially bounded functions mapping from $[a, b]$ into the admissible input set $\mathcal{U}$ (a.e.). In other words, a feasible input generator maps from an initial state inside the set $\mathcal{X}_0$ and a sampling instant t_i to an input for $[t_i, t_{i+1})$ that is measurable, satisfies the input constraints almost everywhere (besides a number of points with measure zero), keeps the state inside of the allowed set of states $\mathcal{X}$, and (at least) renders the set $\mathcal{X}_0$ invariant at the recalculation instants.

Note that Definition 4.1 does not require nor exclude input generators that produce piecewise constant (or in an other form parameterized) inputs (Clarke et al., 2000; Clarke et al., 1997; Ceragioli, 2002; Fontes, 2003). It is rather required that the solution of the differential equation is absolutely continuous for all $x(t_i) \in \mathcal{X}_0$, $t_i \in \pi$ on $[t_i, t_{i+1})$. In the following section we derive general stability conditions for sampled-data open-loop feedback.

4.3 Convergence of Sampled-data Control

In this section we derive conditions for stability of the closed-loop with respect to a set $\mathcal{A} \subseteq \mathcal{X}$. The consideration of a set allows to look at the stabilization problem in a wider context, i.e. one can for example consider the stabilization of orbits, robust stabilization problems (compare Chapter 5), or the stabilization of regions for which the system does not even possess a steady state. Practically, the consideration of a region rather then a fixed set-point is of interest, since often specifications are not given as one fixed, strictly achievable value. Rather they are often given in terms of zones or bands of acceptable performance. Note that the important case of a (single) steady state x_s is contained in the derived result setting $\mathcal{A}=\{x_s\}$.

In the following we denote by $\|x\|_{\mathcal{A}}$ the distance of a point to a set $\mathcal{A}$ defined as follows

Definition 4.2 (Distance to a set)
Given a closed set $\mathcal{A} \subset \mathbb{R}^n$, and a point $x \in \mathbb{R}^n$ we denote

$$\|x\|_{\mathcal{A}} = \inf_{z\in\mathcal{A}} \|x - z\| \tag{4.3}$$

as the distance of the point x to the set $\mathcal{A}$.

We furthermore denote a function as positive definite with respect to a set $\mathcal{A}$ (Yoshizawa, 1966), if

Definition 4.3 (Positive definiteness with respect to a set)
A scalar function $\alpha(x)$ defined for $x \in \mathcal{X}$ is denoted as positive definite with respect to a set $\mathcal{A}$, if $\alpha(x) = 0$ for $x \in \mathcal{A}$ and if for each $\epsilon > 0$ and each compact set $\tilde{\mathcal{X}} \subseteq \mathcal{X}$, there exist positive numbers $\delta(\epsilon, \tilde{\mathcal{X}})$ such that

$$\alpha(x) \geq \delta(\epsilon, \tilde{\mathcal{X}}) \quad \text{for} \quad x \in \tilde{\mathcal{X}}/\mathcal{N}(\epsilon, \mathcal{A}), \tag{4.4}$$

where $\mathcal{N}(\epsilon, \mathcal{A})$ represents the set consisting of $\mathcal{A}$ and it's ϵ neighborhood, i.e. $\mathcal{N}(\epsilon, \mathcal{A}) = \{x \in \mathbb{R}^n | \|x\|_{\mathcal{A}} \leq \epsilon\}$.

The following stability result is closely related to ideas utilized in stabilizing sampled-data open-loop NMPC approaches (Fontes, 2000b; Chen and Allgöwer, 1998b; Jadbabaie et al., 2001). Furthermore, the results are connected to recent results on the link between asymptotic stability and feedback stabilization, see e.g. (Clarke et al., 1997; Marchand and Alamir, 2000; Shim and Teel, 2003). For the proof of the stability result we need the following lemma:

Lemma 4.1
Let $\mathcal{A}$ be a compact set and $\beta : \mathcal{X} \to \mathbb{R}^+$ be a positive definite function with respect to $\mathcal{A}$. Furthermore, let $x(\cdot) : \mathbb{R}^+ \to \mathcal{X}$ be an absolutely continuous function with $\|\dot{x}(\cdot)\|_{\mathcal{L}^\infty(0,\infty)} < \infty$ and:

$$\lim_{T\to\infty} \int_0^T \beta(x(s))ds < \infty, \quad \|x(\cdot)\|_{\mathcal{L}^\infty(0,\infty)} < \infty \ . \tag{4.5}$$

Then $\|x\|_{\mathcal{A}} \to 0$ as $t \to \infty$.

Proof: The proof can be found in Appendix A. ∎

Theorem 4.1 *(***Convergence***)*
Assume that Assumption 4.1 holds. Given a partition π, a compact set $\mathcal{A} \subset \mathbb{R}^n$, and sets $\mathcal{U} \subseteq \mathbb{R}^m$, $\mathcal{X}_0$, $\mathcal{X}$ with $\mathcal{A} \subseteq \mathcal{X}_0 \subseteq \mathcal{X} \subseteq \mathbb{R}^n$. Assume that the input generator u_{SD} is admissible and that there exist,

with respect to the set $\mathcal{A}$, positive definite functions $\beta : \mathcal{X} \to \mathbb{R}^+$ and $\alpha : \mathcal{X} \to \mathbb{R}^+$ such that for all $t_i \in \pi$, $x(t_i) \in \mathcal{X}_0$ and all $\tau \in [t_i, t_{i+1})$

$$\alpha(x(t_i + \tau; x(t_i), u_{SD}(\cdot; x(t_i), t_i))) - \alpha(x(t_i)) \leq -\int_{t_i}^{t_i+\tau} \beta(x(s; x(t_i), u_{SD}(\cdot; x(t_i), t_i)))ds \quad (4.6)$$

and

$$\alpha(x(t_{i+1}) - \alpha(x(t_i)) \leq -\int_{t_i}^{t_{i+1}} \beta(x(s; x(t_i), u_{SD}(\cdot; x(t_i), t_i)))ds \quad (4.7)$$

holds. Then for all $x(0) \in \mathcal{X}_0$: 1.) The solution of (4.1) *subject to* (4.2) *exists for all times. 2.) The input and state constraints are satisfied. 3.) $x(t_i) \in \mathcal{X}_0 \; \forall t_i \in \pi$. 4.) $\|x(t)\|_{\mathcal{A}} \to 0$ as $t \to \infty$.*

Proof: Since $x(0) \in \mathcal{X}_0$ the input $u_{SD}(\cdot; x(0), 0, t_1)$ is admissible, $x(t_1) \in \mathcal{X}_0$, and $x(\tau; x_0, u_{SD}(\cdot; x(0), 0))$ exists over $\tau \in [0, t_1]$ and satisfies the state constraints. Thus, since $x(t_1) \in \mathcal{X}_0$ also the input $u_{SD}(\cdot; x(t_1), t_1, t_2)$ at time t_1 exists and is admissible. Repeatedly applying this argument establishes part 1.)-3.)x of the theorem. The x trajectory resulting from the concatenation is absolutely continuous since the sub-pieces are absolutely continuous. Furthermore, since $u_{SD}(\cdot; x, t_i) \in \mathcal{L}^\infty([t_i, t_{i+1}], \mathcal{U})$

$$\|\dot{x}(\cdot)\|_{\mathcal{L}^\infty(0,\infty)} < \infty \text{ and } \|x(\cdot)\|_{\mathcal{L}^\infty(0,\infty)} < \infty. \quad (4.8)$$

Note that 1.)-3.) (4.6) and (4.7) imply that

$$\forall t \geq 0: \quad \alpha(x(t)) - \alpha(x(0)) \leq -\int_0^t \beta(x(s))ds \quad (4.9)$$

where $x(t)$ denotes the solution of (4.1) starting from $x(0)$. Thus

$$0 \leq \alpha(x(t)) \leq \alpha(x(0)) - \int_0^t \beta(x(s))ds. \quad (4.10)$$

Since $\alpha(x(0)) > 0$, α positive definite with respect to $\mathcal{A}$, and $x(t)$ is bounded, we conclude that $t \to \int_0^t \beta(x(s))ds$ is bounded, i.e.

$$\lim_{T\to\infty} \int_0^T \beta(x(s)))ds < \infty. \quad (4.11)$$

Theorem 4.1 follows now directly from the application of Lemma 4.1. ∎

Condition (4.6) and (4.7) can be seen as contraction requirements. Condition (4.6) implies the decrease of the function α in between recalculation instants of the trajectory generator u_{SD}, whereas (4.7) enforces a decrease from recalculation instant to recalculation instant. Note however, that Condition (4.6) and (4.7) do not imply that the Lyapunov/decreasing function α itself is continuous or strictly decreasing along solution trajectories (compare Figure 4.4). This is important, since it allows to apply Theorem 4.1 to problems that do not admit a Lyapunov function which is continuous in the state. Furthermore, allowing for discontinuity is advantageous in the case of NMPC, for which it cannot be guaranteed a priory that the feedback and the value function is continuous, especially if state constraints are present (Meadows et al., 1995; Fontes, 2000a; Fontes, 2003; Grimm et al., 2003a; Grimm et al., 2003b; Findeisen et al., 2003e).

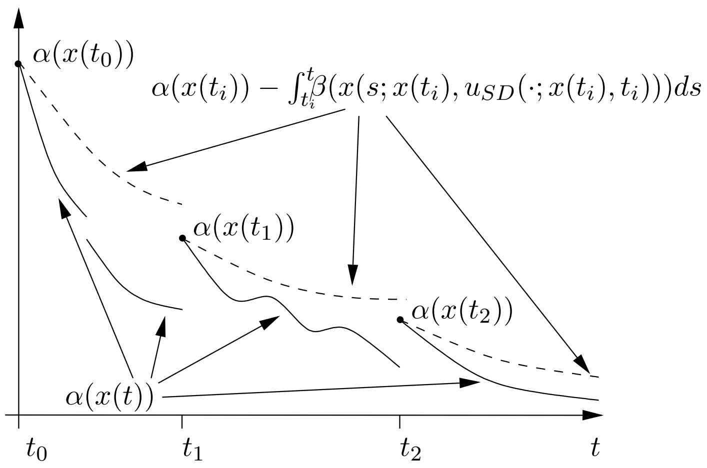


Figure 4.4: Decreasing property of α.

Remark 4.2 *(Relation to existing results) In comparison to the results in (Clarke et al., 1997; Marchand and Alamir, 2000; Shim and Teel, 2003) we do not require that the input is kept constant in between recalculation instants. This allows to establish attractiveness of the set $\mathcal{A}$ without the necessity to decrease the sampling/recalculation time to zero. Our results differ from the results of (Marchand and Alamir, 1998) in that we do not assume an instantaneous feedback, whereas in comparison to (Alamir and Bonard, 1999) we do not assume that the input trajectory generator can steer the system in finite time to the origin.*

Remark 4.3 *(Stability in the sense of Lyapunov) The derived results only imply stability in the sense of convergence to the set $\mathcal{A}$. However, often stability in the sense of Lyapunov is of specific interest. Considering a time invariant autonomous system*

$$\dot{x} = \tilde{f}(x) \tag{4.12}$$

stability in the sense of Lyapunov with respect to a set $\mathcal{A}$ is defined as:

Definition 4.4 (Stability with respect to a set (Yoshizawa, 1966))
A time invariant nonlinear system (4.12) *is stable with respect to a set $\mathcal{A}$, if for all $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$, such that $\|x(t_0)\|_{\mathcal{A}} < \delta(\epsilon) \Rightarrow \|x(t)\|_{\mathcal{A}} < \epsilon, \ \ \forall t \geq t_0$.*

The system is furthermore denoted as asymptotically stable with respect to the set $\mathcal{A}$, if

Definition 4.5 (Asymptotic stability with respect to a set (Yoshizawa, 1966))
A time invariant nonlinear system (4.12) *is asymptotically stable with respect to a set $\mathcal{A}$, if for all $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$, such that $\|x(t_0)\|_{\mathcal{A}} < \delta(\epsilon) \Rightarrow \|x(t)\|_{\mathcal{A}} < \epsilon, \ \ \forall t \geq t_0$ and $\|x(t)\|_{\mathcal{A}} \to 0$ as $t \to \infty$.*

These stability conditions are, however, not directly applicable for systems under sampled-data open-loop feedback, since the input applied in between the recalculation instants t_i and t_{i+1} is given by the system state at time $x(t_i)$:

$$\dot{x}(t) = f(x(t), u_{SD}(t; x(t_i), t_i)), \quad x(0) = x_0. \tag{4.13}$$

Thus, the system possesses a "discrete" memory, $x(t_i)$ that must be taken into account, i.e. the behavior of the system is not only defined by the current state (and possibly time) as assumed in the standard notion of Lyapunov stability. Rigorously one must consider the stability of a hybrid system (Michel, 1999; Grossman et al., 1993; Ye et al., 1998; Hou et al., 1997) consisting of the "discrete" state $x(t_i)$, the continuous state $x(t)$ and a generalized time consisting of the continuous time t and the discrete time instants t_{i+1}.

We do not give explicit conditions for achieving stability in the sense of Lyapunov. The main reason for this is that it requires rather strong conditions on the "value" function α and on the decrease function β.

We outline in Section 4.4 two sampled-data open-loop feedback approaches that satisfy the conditions of Theorem 4.1.

4.3.1 Expansions and Generalizations

Various expansions of the derived results are possible. We shortly outline some of them:

Explicit dependence of the decrease function β on the input:
The integrand on the right hand side of (4.6) can also explicitly depend on the input u_{SD}, i.e., it is possible to utilize a positive function $\tilde{\beta}(x, u)$ as integrand. However, to establish convergence, it must be possible to bound $\tilde{\beta}$ from below via a function $\beta(x)$, positive definite with respect to the set $\mathcal{A}$. This is utilized in the case of sampled-data open-loop NMPC.

Dynamic feedbacks:
Theorem 4.1 is not strictly limited to static input generators, i.e. to input generators that only depend on $x(t_i)$. If, for example, the input generator u_{SD} itself not only depends on $x(t_i)$ but also on an internal state, i.e. $u(t) = u_{SD}(t; x(t_i), x_{SD}(t_i), t_i)$, where x_{SD} is given by

$$\dot{x}_{SD} = f_{SD}(x, x_{SD}, u_{SD}), \tag{4.14}$$

the results of Theorem 4.1 can be applied if one considers the expanded state vector $\tilde{x} = [x, x_{SD}]^T$ instead of x in Theorem 4.1. This is, for example of interest for outputfeedback considerations, see Chapter 6. Further applications include the usage of exogenous disturbance models for tracking and disturbance rejection following the ideas presented in (Isidori, 1995).

Finite time convergence to $\mathcal{A}$:
Often it is of interest to achieve finite time convergence to a set $\mathcal{B} \supset \mathcal{A}$. Theorem 4.1 can be modified to cover this case. However, for this we have to strengthen the assumptions on α slightly.

Corollary 4.1 **(Finite time convergence to a set $\mathcal{B} \supset \mathcal{A}$)**
Assume that the assumptions of Theorem 4.1 hold. Furthermore, assume that α is finite for all $x \in \mathcal{X}$ and that for the compact set $\mathcal{B}$ with $\mathcal{A} \subset \mathcal{B} \subseteq \mathcal{X}_0$ there exists a positive constant $a_{\mathcal{B}}$ such that for all $x \in \mathcal{X}/\mathcal{B}$

$$\alpha(x) > a_{\mathcal{B}}. \tag{4.15}$$

Then, additionally to the result of Theorem 4.1, there exists a recalculation time $t_{conv} \in \pi$ such that $\|x(t)\| \in \mathcal{B}$ for $t \geq t_{conv}$.

Proof: Since $\mathcal{A}$ is a strict subset of $\mathcal{B}$, $\beta(x)$ positive definite with respect to $\mathcal{A}$ and x absolutely continuous, we know that there exists for all $x(t_i) \notin \mathcal{B}$ a finite constant $\Delta\alpha > 0$, such that

$$\alpha(x(t_{i+1})) - \alpha(x(t_i)) \leq -\int_{t_i}^{t_{i+1}} \beta(x(s;x(t_i),u_{SD}(\tau;x(t_i),t_i)))ds \leq -\Delta\alpha. \tag{4.16}$$

Thus, α is strictly decreasing from recalculation instant to recalculation instant, at least as long as $x(t_i) \notin \mathcal{B}$. Furthermore, since α is strictly larger then 0 for all $x \in \mathcal{X}_0/\mathcal{B}$, we know that there must exist a recalculation instant $t_{conv} \in \pi$ such that $\alpha(x(t_{conv})) \leq a_{\mathcal{B}}$. Due to (4.15) this implies that $\alpha(x(t_{conv})) \in \mathcal{B}$. Additionally, we know from (4.6) and (4.7) that the value of $\alpha(x(t))$ for $t \geq t_{conv})$ will be always less or equal to $\alpha(x(t_{conv}))$, thus $x(t) \in \mathcal{B}$ for all $t \geq t_{conv}$. ∎

A variant of this result is utilized for obtaining inherent robustness and output-feedback results in Chapter 5 and Chapter 6.

4.4 Suitable Sampled-data Feedbacks

The conditions for an application of Theorem 4.1 are, on a first view, rather stringent and difficult to satisfy. Nevertheless, the results can be applied in a series of cases. Examples are the approaches outlined in (Alamir and Bonard, 1999; Marchand and Alamir, 1998), sampled-data predictive control, and for example open-loop input generators based on differential flatness. In the following we consider the application of the result to two specific sampled-data open-loop feedbacks in more detail. Firstly we consider the application of continuous instantaneous feedbacks to achieve sampled-data control via open-loop forward simulation. Secondly in Section 4.4.2 we apply the derived result to obtain stability conditions for sampled-data open-loop predictive control.

4.4.1 Instantaneous Feedbacks and Sampled-data Control

In the following we show that Theorem 4.1 allows to derive that any asymptotically stabilizing locally Lipschitz continuous, instantaneous feedback can be used to obtain a stabilizing sampled-data open-loop feedback.

For simplicity of derivation we only consider the stabilization of the origin, i.e. $\mathcal{A} = \{0\}$ and $f(0,0) = 0$. We furthermore assume that f is locally Lipschitz and that no constraints on the states and inputs are present, i.e. $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{U} = \mathbb{R}^m$.

Assume that a locally Lipschitz continuous state-feedback $k : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is known that asymptotically stabilizes the equilibrium of the closed-loop system

$$\dot{x} = f(x, k(x)) \tag{4.17}$$

with a (non-zero) region of attraction $\mathcal{R} \subseteq \mathbb{R}^n$. Based on this state-feedback controller one can obtain a stabilizing open-loop input trajectory generator by simple forward integration, i.e. u_{SD} is given by

$$u_{SD}(\tau; x(t_i), t_i) = k(\bar{x}(\tau)), \tau \in [t_i, t_{i+1}), \tag{4.18}$$

where $\bar{x}$ is the solution trajectory of

$$\dot{\bar{x}}(\tau) = f(\bar{x}(\tau), k(\bar{x}(\tau))) \text{ with } \bar{x}(t_i) = x(t_i). \tag{4.19}$$

In other words the input signal applied open-loop in between $[t_i, t_{i+1})$ is simply given by a forward simulation of the closed-loop system. Reasons for applying such an approach might be measurements that are only available at the recalculation times. For the input u_{SD} given by (4.18)- (4.19) it is straightforward to derive the following result:

Theorem 4.2 *(***Nominal convergence by feedforward simulation of an instantaneous feedback***)* *Assume that f is locally Lipschitz continuous, that $f(0,0) = 0$, that there are no constraints on the state or inputs, and that there are no disturbances present. Then for any instantaneous locally Lipschitz continuous state-feedback $k : \mathbb{R}^n \rightarrow \mathbb{R}^m$ that asymptotically stabilizes the equilibrium of the closed-loop system $\dot{x} = f(x, k(x))$ with a region of attraction $\mathcal{R}$ that is applied in a sampled-data open-loop fashion as defined in* (4.18)- (4.19) *π, with $\bar{\pi}$ finite, for all $x(0) \in \mathcal{R}$ the solution of* (4.1) *exists for all times and $\|x(t)\| \rightarrow 0$ as $t \rightarrow \infty$.*

Proof: We know that the origin of $\dot{\bar{x}} = f(\bar{x}, k(\bar{x}))$ is asymptotically stable. Then, the converse Lyapunov theorem given in (Kurzweil, 1956) assures the existence of a continuously differentiable $\mathcal{K}$-function $\alpha(\bar{x})$ and three $\mathcal{K}$-functions $\gamma_1(\bar{x})$, $\gamma_2(\bar{x})$, and $\beta(\bar{x})$ defined on the region of attraction $\mathcal{R}$, such that

$$\gamma_1(\bar{x}) \leq \alpha(\bar{x}) \leq \gamma_2(\bar{x}) \tag{4.20}$$

$$\frac{\partial \alpha}{\partial \bar{x}} f(\bar{x}(\tau), k(\bar{x}(\tau))) \leq -\beta(\bar{x}). \tag{4.21}$$

Since we consider the nominal case we furthermore know that x and $\bar{x}$ coincide starting from the same $x(t_i)$ applying u_{SD} (which is given in terms of $\bar{x}$). Integrating (4.21) over $[t_i, t_{i+1})$ replacing $\bar{x}$ by x we see that condition (4.6) of Theorem 4.1 is satisfied. Furthermore, due to continuity of the trajectories from t_{i+1}^- to t_{i+1} we know that condition (4.7) holds. Thus, the conditions for the application of Theorem (4.6) hold and one obtains the stated result. ■

The theorem implies that any locally Lipschitz continuous asymptotically stabilizing instantaneous feedback can be modified to achieve a nominally, in the sense of convergence stabilizing sampled-data open-loop feedback. This provides a simple way for obtaining a sampled-data feedback controller by forward simulation of the nominal closed-loop system equations.

Remark 4.4 *(Nominal convergence) Note that the given result is mainly of theoretical value and does not imply any stability or performance results for the disturbance case. The achievable performance and stability in this case will be discussed in the next chapter.*

Remark 4.5 (Varying recalculation times) *Note that the derived result, in the nominal case, does not require any conditions on the partition of the recalculation instants. In principle the partition can be equidistant or varying, depending on the problem considered. This allows to apply this method to problems, where state information might be available only rarely and at varying time instants. This might for example be due to communication restrictions between the controller and sensors, as they are typically present in industrial control implementations working with a central communication bus that is used for multiple control loops. Furthermore, the measurement instants might vary, since they must be performed by a human, such as a chemical analysis of a composition performed in an off-site laboratory. With respect to the recalculation partition, however, note that the maximum recalculation time is closely connected to a possibly tolerable external disturbance and/or model/plant uncertainty. This is outlined in the robustness considerations of the next chapter and the output-feedback considerations in Chapter 6.*

4.4.2 Stability of Sampled-data NMPC

In the following we apply Theorem 4.1 to derive stability conditions for sampled-data open-loop NMPC. The presented results are an expansion of the results presented in (Fontes, 2000b) in the sense that the stabilization with respect to a closed target set $\mathcal{A}$ are considered, and that we require less restrictive conditions to hold.

As in Section 2.5.1 the input applied in between the recalculation instants is given by the solution of the following open-loop optimal control problem:

$$\min_{\bar{u}(\cdot)\in\mathcal{L}^\infty([0,T_p],\mathcal{U})} J(\bar{x}(\cdot),\bar{u}(\cdot)) \tag{4.22a}$$

subject to:

$$\dot{\bar{x}}(\tau)=f(\bar{x}(\tau),\bar{u}(\tau)),\quad \bar{x}(t_i)=x(t_i) \tag{4.22b}$$

$$\bar{x}(\tau)\in\mathcal{X}\quad \tau\in[t_i,t_i+T_p] \tag{4.22c}$$

$$\bar{x}(t_i+T_p)\in\mathcal{E}. \tag{4.22d}$$

As in Chapter 2, the barred variables denote predicted variables. The cost functional J minimized over the control horizon $T_p\geq\bar{\pi}>0$ is given by:

$$J(\bar{x}(\cdot),\bar{u}(\cdot))=\int_{t_i}^{t_i+T_p} F(\bar{x}(\tau),\bar{u}(\tau))d\tau+E(\bar{x}(t_i+T_p)). \tag{4.23}$$

The stage cost $F:\mathcal{X}\times\mathcal{U}\rightarrow\mathbb{R}^+$ is assumed to be positive definite with respect to the closed target set $\mathcal{A}$ independent of $u\in\mathcal{U}$, i.e.

Assumption 4.2 *(***Positive definiteness of the cost function with respect to the set** $\mathcal{A}$*)*
There exists a function $\beta:\mathcal{X}\rightarrow\mathbb{R}^+$, positive definite with respect to the set $\mathcal{A}$, such that

$$F(x,u)\geq\beta(x),\quad\forall x\in\mathcal{X},\ u\in\mathcal{U}. \tag{4.24}$$

As in Chapter 2 the terminal region constraint $\mathcal{E}\supseteq\mathcal{A}$ and the terminal penalty term $E:\mathbb{R}^n\rightarrow\mathbb{R}^+$ are used to enforce stability and to improve the performance of the closed-loop. We assume that E and $\mathcal{E}$ satisfy the following assumptions:

Assumption 4.3 *(***Terminal region***)*
The terminal region $\mathcal{E}\subseteq\mathcal{X}$ is closed with $\mathcal{A}\subseteq\mathcal{E}$ and the terminal penalty term $E(x)$ is continuous and positive semi-definite with respect to the set $\mathcal{A}$.

Assumption 4.4 *(***Existence of a local open-loop input** $u_{\mathcal{E}}(\cdot)$*)*
For all $\tilde{x}\in\mathcal{E}\backslash\mathcal{A}$ there exists an input signal $u_{\mathcal{E}}(\cdot;\tilde{x})\in\mathcal{L}^\infty([0,\bar{\pi}],\mathcal{U})$, such that

$$x(\tau;\tilde{x},u_{\mathcal{E}}(\cdot;\tilde{x}))\in\mathcal{E},\qquad\forall\tau\in[0,\bar{\pi}] \tag{4.25}$$

and

$$E(x(\tau;\tilde{x},u_{\mathcal{E}}(\cdot;\tilde{x})))-E(\tilde{x})\leq-\int_0^\tau F(x(\tau;\tilde{x},u_{\mathcal{E}}(\cdot;\tilde{x})),u_{\mathcal{E}}(s;\tilde{x}))ds,\ \ \forall\tau\in[0,\bar{\pi}]. \tag{4.26}$$

Loosely speaking, similarly to the case of convergence to the origin as presented in Section 2.5.1, E can be seen as a F-conform local control Lyapunov function for the terminal set $\mathcal{E}$. Suitable terminal penalty terms and terminal regions can be determined similarly to the results presented in (Michalska and Mayne, 1993; Chen and Allgöwer, 1998b; Chen and Allgöwer, 1998a; Chen et al., 2000; Magni and Scattolini, 2002; Fontes, 2000b; Fontes, 2003) for the case of convergence to the origin.

The main difference to the existing results that achieve convergence to the origin lies in the fact that one can utilize local controllers that only achieve stabilization with respect to the set $\mathcal{A}$ for calculating $\mathcal{E}$ and E. This expands the applicability of the results significantly, e.g. in the case that only set stabilization is required or can be achieved, as in the output-feedback case (see Chapter 6).

The optimal input resulting from the solution of (4.22) (assuming that it exists, see remark later), is denoted by $\bar{u}^\star(\cdot;x(t_i))$. It defines the open-loop input that is applied to the system until the next sampling instant t_{i+1}:

$$u_{SD}(t;x(t_i),t_i)=\bar{u}^\star(t;x(t_i)),\quad t\in[t_i,t_{i+1})\,, \tag{4.27}$$

i.e. in accordance with Definition 4.1, the admissible input generator is defined via the solution of the optimal control problem (4.22).

We refer in the following to an admissible set of problem (4.22) as:

Definition 4.6 ***(*****Admissible state set** X***)***
A set $X \subseteq \mathcal{X}$ *is called admissible, if for all* $x_0 \in X$ *there exists an admissible input* $u(\cdot) \in \mathcal{L}^\infty([0,T_p],\mathcal{U})$ *such that*

1. $x(\tau;x_0,\tilde{u}(\cdot)) \in X,\ \tau \in [0,T_p]$
2. $x(T_p;x_0,\tilde{u}(\cdot)) \in \mathcal{E}.$

Note that the admissibility of a set $X \subseteq \mathcal{X}$ does not imply that a solution to the optimal control problem (4.22) exists for all $x \in X$. Specifically, it could be that the minima of problem (4.22) is not attained. While the existence of an admissible input is related to constrained controllability, the question about the existence of an optimal solution of (4.22) is in general non-trivial to answer. Thus, we consider in the following a maximum admissible set, such that (4.22) has a solution:

Definition 4.7 ***(*****Set** $\mathcal{R}$***)***
The set $\mathcal{R}$ *denotes the maximum admissible set, such that* (4.22) *admits for all* $x_0 \in \mathcal{R}$ *an optimal (not necessarily unique) solution.*

Note that in general it is rather difficult to explicitly calculate the maximum admissible set (Rossiter et al., 1995; Mayne et al., 2000). The main reason for considering the set $\mathcal{R}$ is the requirement that an optimal/feasible solution at one sampling instant should guarantee the existence of a feasible and optimal solution at the next sampling instant.

Remark 4.6 *(Existence of solutions) It is possible to derive strict existence results for* (4.22) *imposing certain convexity and compactness conditions, see for example (Fontes, 2003; Fontes, 2000b; Michalska and Vinter, 1994) and (Berkovitz, 1974; Fleming and Rishel, 1982; Vinter, 2000). However, often it is not possible to check these conditions a priori. Furthermore, these conditions are often rather restrictive with respect to the allowed system class and cost functional.*

Under the assumptions made, the following theorem states sufficient conditions guaranteeing stability of the closed-loop in the sense of convergence to the set $\mathcal{A}$:

Theorem 4.3 (Convergence of sampled-data NMPC)
Suppose that Assumptions 4.1-4.4 hold. Then for the closed-loop system defined by (4.1), (4.2) *and* (4.27) $\|x(t)\|_{\mathcal{A}} \to 0$ *as* $t \to \infty\ \forall x(0) \in \mathcal{R}$.

Proof: As usual in predictive control the proof consists of two parts: The first part establishes that initial feasibility implies feasibility afterwards. Based on this result it is then shown that the state converges to the set $\mathcal{A}$ considering the value function of (4.22) as a decreasing/Lyapunov like function.
Feasibility:
Consider any recalculation instant t_i for which a solution exists (e.g. t_0). In between t_i and t_{i+1} the optimal input $\bar{u}^\star(\tau;x(t_i))$ is implemented. Since no model-plant mismatch nor disturbances are present, $x(t_{i+1}) = \bar{x}(t_{i+1};x(t_i),\bar{u}^\star(\tau;x(t_i)))$. Thus, the remaining piece of the optimal input $\bar{u}^\star(\tau;x(t_i)),\ \tau \in$

$[t_{i+1}, t_i + T_p]$ satisfies the state and input constraints. Furthermore, $\bar{x}(t_i+T_p; x(t_i), \bar{u}^\star(\tau; x(t_i))) \in \mathcal{E}$. Thus, it follows from Assumption 4.4 that there exists at least one input $u_{\mathcal{E}}(\cdot; x)$ that renders $\mathcal{E}$ invariant for the predicted state $\bar{x}$ over $[t_i + T_p, t_i + T_p + \bar{\pi}]$. Picking any such input we obtain as admissible input for $t_i + \sigma$, $\sigma \in (0, t_{i+1} - t_i]$

$$\tilde{u}(\tau; x(t_i + \sigma)) = \begin{cases} \bar{u}^\star(\tau; x(t_i)), \ \tau \in [t_i + \sigma, t_i + T_p] \\ u_{\mathcal{E}}(\tau - t_i - T_p), \tau \in (t_i + T_p, t_i + T_p + \sigma] \end{cases}. \tag{4.28}$$

Specifically, we have for the next recalculation instant ($\sigma = t_{i+1} - t_i$) that $\tilde{u}(\cdot; x(t_{i+1}))$ is an admissible input, hence admissibility at time t_i implies admissibility at t_{i+1}. Thus, if (4.22) is feasible for $t = 0$, it is feasible for all $t \geq 0$.

Convergence:
We first show that the value function is decreasing starting from a sampling instant. Remember that the value of the value function V for $x(t_i)$ is given by:

$$V(x(t_i)) = \int_{t_i}^{t_i+T_p} F(\bar{x}(\tau; x(t_i), \bar{u}^\star(\cdot; x(t_i))), \bar{u}^\star(\tau; x(t_i)))d\tau + E(\bar{x}(t_i + T_p; x(t_i), \bar{u}^\star(\cdot; x(t_i)))), \tag{4.29}$$

and the cost resulting from (4.28) starting from any $x(t_i + \sigma; x(t_i), \bar{u}^\star(\cdot; x(t_i)))$, $\sigma \in (0, t_{i+1} - t_i]$, using the input $\tilde{u}(\tau, x(t_i + \sigma))$, is given by:

$$J(x(t_i + \sigma), \tilde{u}(\cdot; x(t_i + \sigma))) = \int_{t_i+\sigma}^{t_i+\sigma+T_p} F(\bar{x}(\tau; x(t_i + \sigma), \tilde{u}(\cdot; x(t_i + \sigma))), \tilde{u}(\tau; x(t_i + \sigma)))d\tau + E(\bar{x}(t_i + \sigma + T_p; x(t_i + \sigma), \tilde{u}(\cdot; x(t_i + \sigma)))). \tag{4.30}$$

Reformulation yields

$$\begin{aligned} J(x(t_i + \sigma), \tilde{u}(\cdot; x(t_i + \sigma))) &= V(x(t_i)) \\ &- \int_{t_i}^{t_i+\sigma} F(\bar{x}(\tau; x(t_i), \bar{u}^\star(\cdot; x(t_i))), \bar{u}^\star(\tau; x(t_i)))d\tau - E(\bar{x}(t_i + T_p; x(t_i), \bar{u}^\star(\cdot; x(t_i)))) \\ &+ \int_{t_i+T_p}^{t_i+\sigma+T_p} F(\bar{x}(\tau; x(t_i + \sigma), \tilde{u}(\cdot; x(t_i + \sigma))), \tilde{u}(\tau; x(t_i + \sigma)))d\tau \\ &+ E(\bar{x}(t_i + \sigma + T_p; x(t_i + \sigma), \tilde{u}(\cdot, x(t_i + \sigma)))). \end{aligned} \tag{4.31}$$

Integrating inequality (4.26) from $t_i+\sigma$ to $t_i+\sigma+T_p$ starting from $x(t_i + \sigma)$ we obtain zero as an upper bound for the last three terms on the right side. Thus,

$$J(x(t_i + \sigma), \tilde{u}(\cdot, x(t_i + \sigma))) - V(x(t_i)) \leq -\int_{t_i}^{t_i+\sigma} F(\bar{x}(\tau; x(t_i), \bar{u}^\star(\cdot; x(t_i))), \bar{u}^\star(\tau; x(t_i)))d\tau. \tag{4.32}$$

Since $\tilde{u}$ is only a feasible but not necessarily the optimal input for $x(t_i + \sigma)$ and since $x_0 \in \mathcal{R}$, it follows that

$$V(x(t_i + \sigma)) - V(x(t_i)) \leq -\int_{t_i}^{t_i+\sigma} F(\bar{x}(\tau; x(t_i), \bar{u}^\star(\cdot; x(t_i))), \bar{u}^\star(\tau; x(t_i)))d\tau. \tag{4.33}$$

Furthermore, since no model-plant mismatch or disturbances are present, we can replace the predicted state $\bar{x}$ and $\bar{u}^{\star}$ on the right side by the real system state x and u_{SD}:

$$V(x(t_i+\sigma))-V(x(t_i)) \leq -\int_{t_i}^{t_i+\sigma} \beta(x(\tau;x(t_i),u_{SD}(\tau;x(t_i),t_i)))d\tau, \tag{4.34}$$

where β is the positive definite function bounding F from below, compare Assumption 4.2. Thus, the value function is decreasing along solution trajectories starting at a sampling instant t_i. Especially we have

$$V(x(t_{i+1}))-V(x(t_i)) \leq -\int_{t_i}^{t_{i+1}} \beta(x(\tau;x(t_i),u_{SD}(\tau;x(t_i),t_i)))d\tau. \tag{4.35}$$

Note that the value function V and the integrand of the right hand side of (4.35) satisfies all required conditions on α and β of Theorem 4.1 considering the set $\mathcal{R}$ as set X_0. Thus we obtain that $\|x(t)\|_{\mathcal{A}} \rightarrow 0$ for $t \rightarrow \infty$ $\forall x(0) \in X_0$. ■

Remark 4.7 *(Consideration of a limited class of input signals/quantized control) The derived result can easily be expanded to allow the consideration of limited classes of input signals instead of a measurable, "constraint satisfying" input signals. Examples are the consideration of inputs that are piecewise continuous in between sampling instants, that are piecewise constant in between the sampling instants, or that are parameterized as splines or polynomials as a function of time. The consideration of such inputs can be of interest, if only piecewise constant inputs can be implemented on the real system due to slow D/A converters or if a direct solution approach for the solution of the optimal control problem* (4.22)*, as outlined in Section 3.2.1, is employed. With respect to the derived convergence/stability result, only minor modifications are necessary. Specifically, one only has to limit the allowed input signals in Definition 4.6 for the admissible set, and one has to limit the considered inputs in the optimal control problem* (4.22) *itself. Considering such modifications, the results of Theorem 4.3 remain unchanged and no modifications in the proof are necessary. Basically it is even possible to consider only a finite number of discrete values for the input (often referred to as quantized control or hybrid control). Note, that even though NMPC allows in principle to consider restricted classes of input signals, finding a suitable terminal penalty term and terminal region constraint can be rather difficult.*

Remark 4.8 *It is possible to replace the minimization in* (4.22) *by a decreasing condition on the value function. Basically it is necessary to achieve a positive decrease in the value function, i.e. feasibility implies stability assuming that certain conditions hold (Chen and Allgöwer, 1998b; Scokaert et al., 1999; Findeisen, 1997; Findeisen and Rawlings, 1997). Results on the stability under sub-optimal solutions can for example be found in (Scokaert et al., 1999; Findeisen, 1997; Findeisen and Rawlings, 1997; Magni et al., 2003) for discrete time NMPC, in (Chen and Allgöwer, 1998b; Fontes, 2000b; de Oliveira Kothare and Morari, 2000) for sampled-data NMPC, and in (Michalska and Mayne, 1993) for instantaneous NMPC.*

In the next section we apply the outlined sampled-data open-loop feedback strategies to a small example problem, underlining the advantage of a sampled-data open-loop feedback implementation in comparison to sample-and hold implementations.

4.4.3 Control of a CSTR

In this section we study the performance and properties of the presented sampled-data open-loop feedback controllers in comparison to instantaneous feedbacks. For this purpose we consider the control of a continuous stirred tank reactor (CSTR) for an exothermic, irreversible reaction, $A \rightarrow B$. The considered model under the assumption of constant liquid volume takes the following form (Henson and Seborg, 1997; Seborg et al., 1999):

$$\dot{c}_A = \frac{q}{V}(c_{Af} - c_A) - k_0 e^{\frac{-E}{RT}} c_A \tag{4.36}$$

$$\dot{T} = \frac{q}{V}(T_f - T) + \frac{-\Delta H}{\rho C_p} k_0 e^{\frac{-E}{RT}} c_A + \frac{UA}{V \rho C_p}(T_c - T). \tag{4.37}$$

The parameters UA, q, V, c_{Af}, E/R, ρ, k_0, $-\Delta H$, C_p can be found in Table 4.1. The concentration

Table 4.1: Parameters of the CSTR model.

Parameter	Value	Parameter	Value
q	100 L/min	C_{Af}	1 mol/L
T_f	350 K	V	100 L
ρ	1000 g/L	C_p	0.239 J/g·K
$-\Delta H$	$5 \cdot 10^4$ J/mol	$\frac{E}{R}$	8750 K
k_0	$7.2 \cdot 10^{10}$ min^{-1}	UA	$5 \cdot 10^4$ J/min·K

of substance A is denoted by c_A, T is the reactor temperature, and T_c is the manipulated variable – the coolant stream temperature. The objective is to stabilize the operating point $T_s = 375K$, $c_{As} = 0.159$mol/L via the coolant stream temperature T_c ($T_{cs} = 302.84K$), while the coolant stream temperature T_c is limited to values between

$$T_c \in [270\text{K}, 330\text{K}]. \tag{4.38}$$

We assume that the state information is only available all 0.15min. This also defines the recalculation instants considered for sampled-data open-loop feedback, i.e. we consider that the recalculation instants are equally apart:

$$t_i = i\delta^r, \quad \text{where } \delta^r = 0.15\text{min}. \tag{4.39}$$

With respect to the manipulated input, the coolant stream temperature, we assume that possibly present D/A converters or other sample-and-hold elements are sufficiently fast and can be neglected.

As outlined in (Henson and Seborg, 1997), the reactor shows significant nonlinearities and varying time constants, see Figure 4.5. The variability of the process dynamics, especially of the time constants, can lead to significant performance decreases or even instability if a sampled-data implementation with a constant input in between the recalculation instants is employed. This is one of the issues examined in the following.

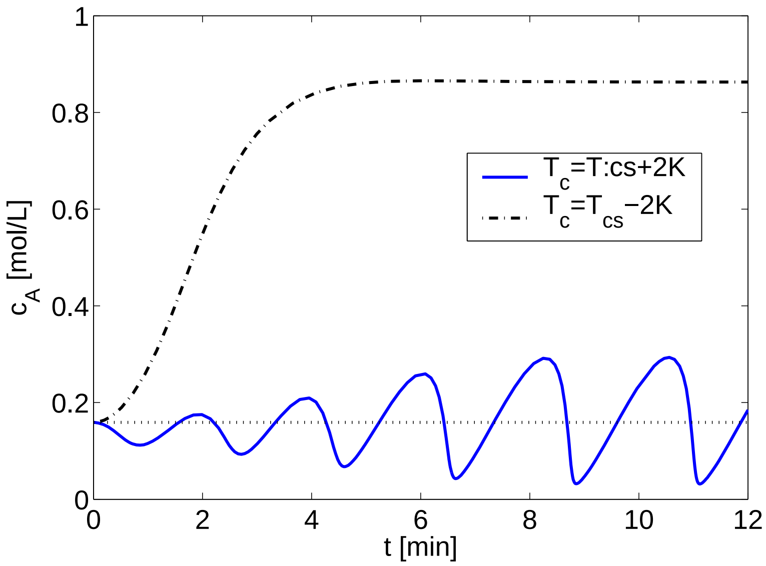


Figure 4.5: Temperature trajectories for ±2K changes in the coolant stream temperature.

Sampled-data open-loop feedback based on a I/O linearizing controller

We apply the feedforward simulation technique outlined in Section 4.4.1 to achieve sampled-data open-loop feedback. The sampled-data open-loop feedback is based on a classical input-output (I/O) linearizing state-feedback controller (Isidori, 1995), neglecting the constraints (4.38) on the manipulated input T_c. Considering as output the reactor temperature T, i.e.

$$y(t) = T(t), \tag{4.40}$$

one trivially sees that the system has relative degree one, and that the zero dynamics is globally exponentially stable. Simply clipping the input once the input constraints (4.38) are violated leads to the following controller:

$$T_c = \begin{cases} 330\text{K} & \text{if } T_{c,\text{iolin}}(T, c_A) > 330\text{K} \\ T_{c,\text{iolin}}(T, c_A) & \\ 240\text{K} & \text{if } T_{c,\text{iolin}}(T, c_A) < 270\text{K} \end{cases}. \tag{4.41}$$

Here $T_{c,\text{iolin}}(T, c_A)$ is the I/O linearizing controller given by:

$$T_{c,\text{iolin}}(T, c_A) = T - \frac{V\rho C_p}{UA}\left(\frac{q}{V}(T_f - T) + \frac{-\Delta H}{\rho C_p} k_0 e^{\frac{-E}{RT}} C_A + \lambda_{\text{iolin}}(T - T_s)\right), \tag{4.42}$$

where T_{cs}, c_{As}, and T_{cs} are the steady state values of the considered operating point, and where $-\lambda_{\text{iolin}}$ defines the closed-loop eigenvalue of the linearized dynamics. In the following $\lambda_{\text{iolin}} = 3$ is used.

This controller stabilizes the system with a rather large region of attraction and satisfies the conditions on the instantaneous feedback of Section 4.4.1. Figure 4.6 shows the performance of the controller for different implementations: for a sampled-data open-loop implementation via feedforward simulation as proposed in Section 4.4.1, a sampled-data implementation considering that the input at the recalculation time t_i is kept constant until the next recalculation time (sampled-and-hold element with a sampling time of $\delta^S = \delta^r$ present), and, for comparison, the instantaneous implementation of (4.42). All simulations where performed for the initial conditions $c_A(0) = 0.5\text{mol/L}$ and $T(0) = 350\text{K}$. The

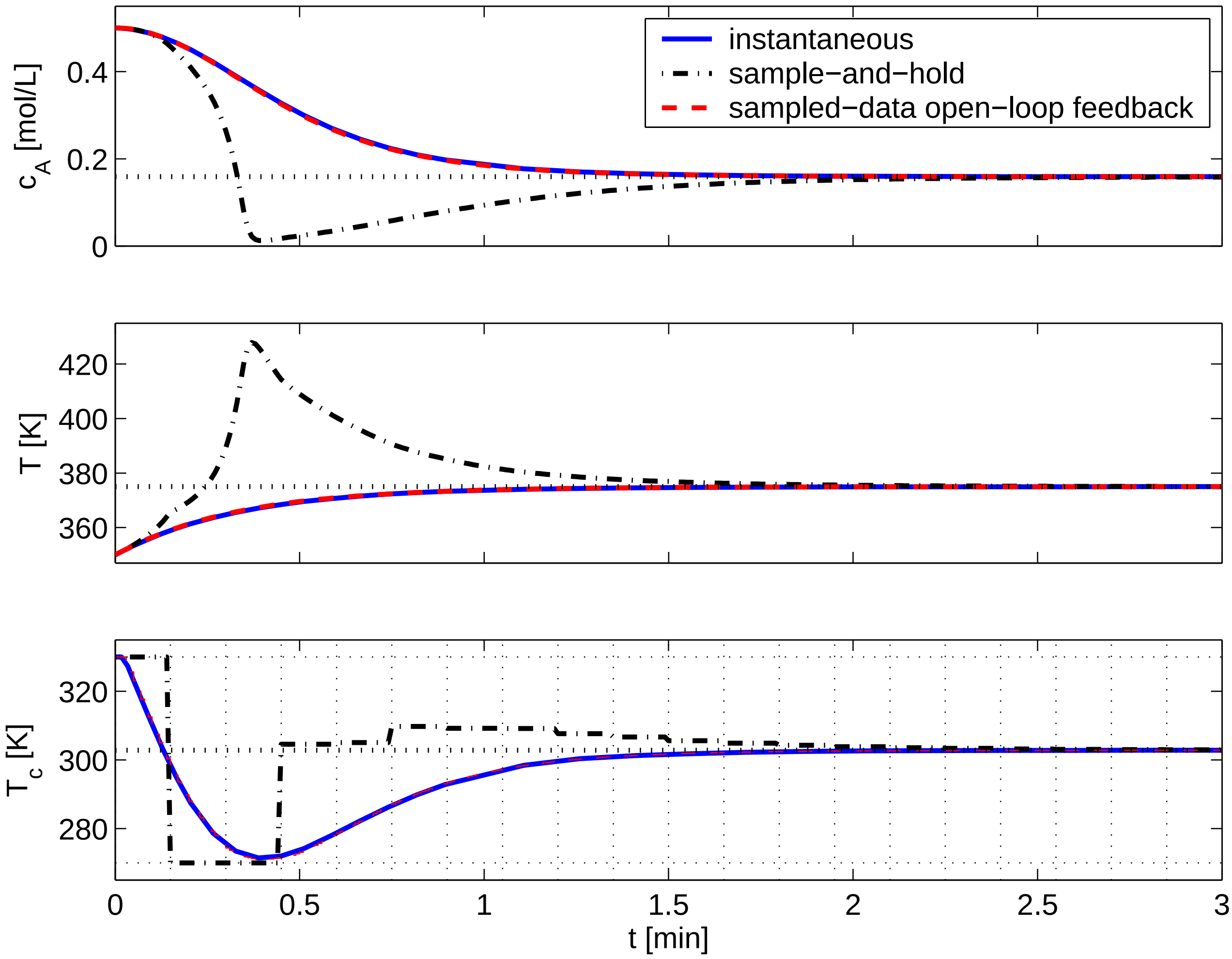


Figure 4.6: States and manipulated input using an instantaneous, a sampled-data open-loop feedback, and a sample-and-hold implementation of an I/O linearizing controller.

sampled-data open-loop feedback is performed via a forward simulation of the closed-loop system in parallel to the real system, compare equation (4.19).

As expected, in the absence of disturbances and model-plant mismatch, the instantaneous and sampled-data open-loop feedback achieve the same performance, which is superior to the performance of the controller considering a sample-and-hold element. To also achieve a good performance with the sample-and-hold element present, one would have to decrease the recalculation time significantly. The presented results make immediately clear, why the consideration of an open-loop input trajectory (or approximations hereof) is advantageous in comparison to an implementation using a sample-and-hold element.

However, the performance of the sampled-data open-loop feedback will in general decrease significantly in the presence of disturbances and model-plant mismatch, as will be outlined in the next chapter. One reason for this is that updated state information is only fed back at the recalculation instants, i.e. no immediate reaction of the controller in between the recalculation to disturbances is possible.

Sampled-data open-loop NMPC

In the following we apply the stabilizing sampled-data open-loop NMPC strategy as outlined in Section 4.4.2 to the CSTR example process. The considered stage cost function F is quadratic and takes the form:

$$F = \begin{bmatrix} c_A - c_{As} \\ T - T_s \end{bmatrix}^T \begin{bmatrix} 40 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} c_A - c_{As} \\ T - T_s \end{bmatrix} + (T_c - T_{cs})^2. \tag{4.43}$$

A suitable terminal penalty term E, and a terminal region $\mathcal{E}$ satisfying Assumption 4.4 are determined on the basis of a locally stabilizing linear control law following the semi-infinite optimization approach as outlined in (Chen and Allgöwer, 1998b). For this purpose the system is linearized around the considered steady state and a linear LQR controller is designed, considering the stage cost (4.43). For the considered steady state we obtain the following linear feedback law

$$T_c = \underbrace{\begin{bmatrix} 352.8246 \\ 4.3889 \end{bmatrix}^T}_{K} \begin{bmatrix} c_A - c_{As} \\ T - T_s \end{bmatrix} + T_{cs} \tag{4.44}$$

The terminal penalty term E and the terminal region $\mathcal{E}$ are determined according to (Chen and Allgöwer, 1998b) based on this linear control law, such that the terminal region $\mathcal{E}$ is maximized in size subject to the conditions, that $\mathcal{E}$ is invariant for the nonlinear system under the local controller, that the linear feedback (4.44) satisfies the input constraints, and that Assumption 4.4 is satisfied, replacing $u_{\mathcal{E}}$ by the linear feedback law. The necessary semi-infinite optimization to obtain a maximum region $\mathcal{E}$ was performed in Matlab, leading to the quadratic terminal penalty

$$E\left(\begin{bmatrix} c_A - c_{As} \\ T - T_s \end{bmatrix}\right) = \begin{bmatrix} c_A - c_{As} \\ T - T_s \end{bmatrix}^T \underbrace{\begin{bmatrix} 19789.05 & 168.34 \\ 168.34 & 0.21 \end{bmatrix}}_{E_L} \begin{bmatrix} c_A - c_{As} \\ T - T_s \end{bmatrix}. \tag{4.45}$$

The terminal region $\mathcal{E}$ is given by

$$\mathcal{E} = \left\{ \begin{bmatrix} c_A \\ T \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, \begin{bmatrix} c_A - c_{As} \\ T - T_s \end{bmatrix}^T E_L \begin{bmatrix} c_A - c_{As} \\ T - T_s \end{bmatrix} \leq \alpha_{E_L} \right\}. \tag{4.46}$$

For the simulations the resulting optimal control problem (4.22) that must be solved at all recalculation instants was solved by a direct solution approach. For this purpose the input was discretized as piecewise constant, with a discretization time $\delta^u = 0.01$min. The prediction horizon was set for all simulations to $T_p = 1.5$min, which leads to a rather large region of attraction of the closed-loop. For the simulations it was assumed that the numerical solution of the optimal control problem can be performed instantaneously, i.e. computational delays were neglected. Figure 4.7 and 4.8 show the

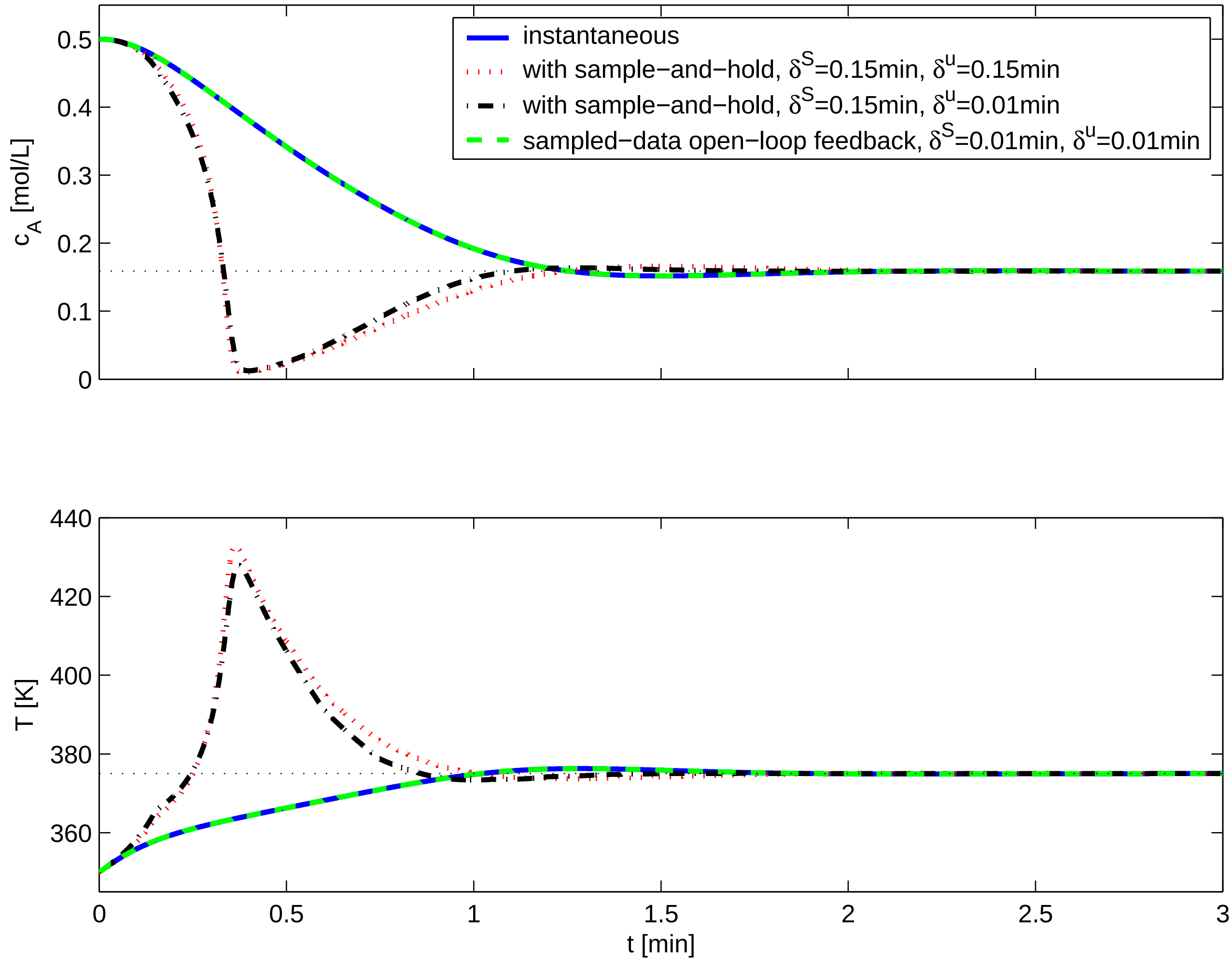


Figure 4.7: Reactor temperature and concentration of the substance A in the reactor for instantaneous, sample-and-hold, and sampled-data open-loop feedback implementations. For the sampled-data and sampled-data open-loop feedback implementation the recalculation time is fixed to $\delta^r = 0.15\text{min}$.

states and inputs of the resulting closed-loop in comparison to an instantaneous NMPC implementation, and two NMPC implementations where the applied input was kept constant in between the recalculation instants (sample-and-hold element present). The simulations where performed for the initial conditions $c_A(0) = 0.5\text{mol/L}$ and $T(0) = 350\text{K}$.

For the instantaneous NMPC implementation, the open-loop optimal control problem was solved at all function calls of the numerical integration algorithm. For the sampled-data open-loop feedback implementation the optimal control problem was solved at all recalculation instants, i.e. all 0.15min and the resulting input was applied open-loop until the next recalculation instant. Even so that a small discretization of the input due to the direct solution approach of the optimal control problem is present, as expected the instantaneous and sampled-data open-loop NMPC implementation show similar performance. In comparison, the sample-and-hold implementations, e.g. fixing the input in between recalculation instants, leads to a significant performance loss. If the sample-and-hold element is not considered in the optimal control problem, e.g. if the input discretization is not adjusted to the

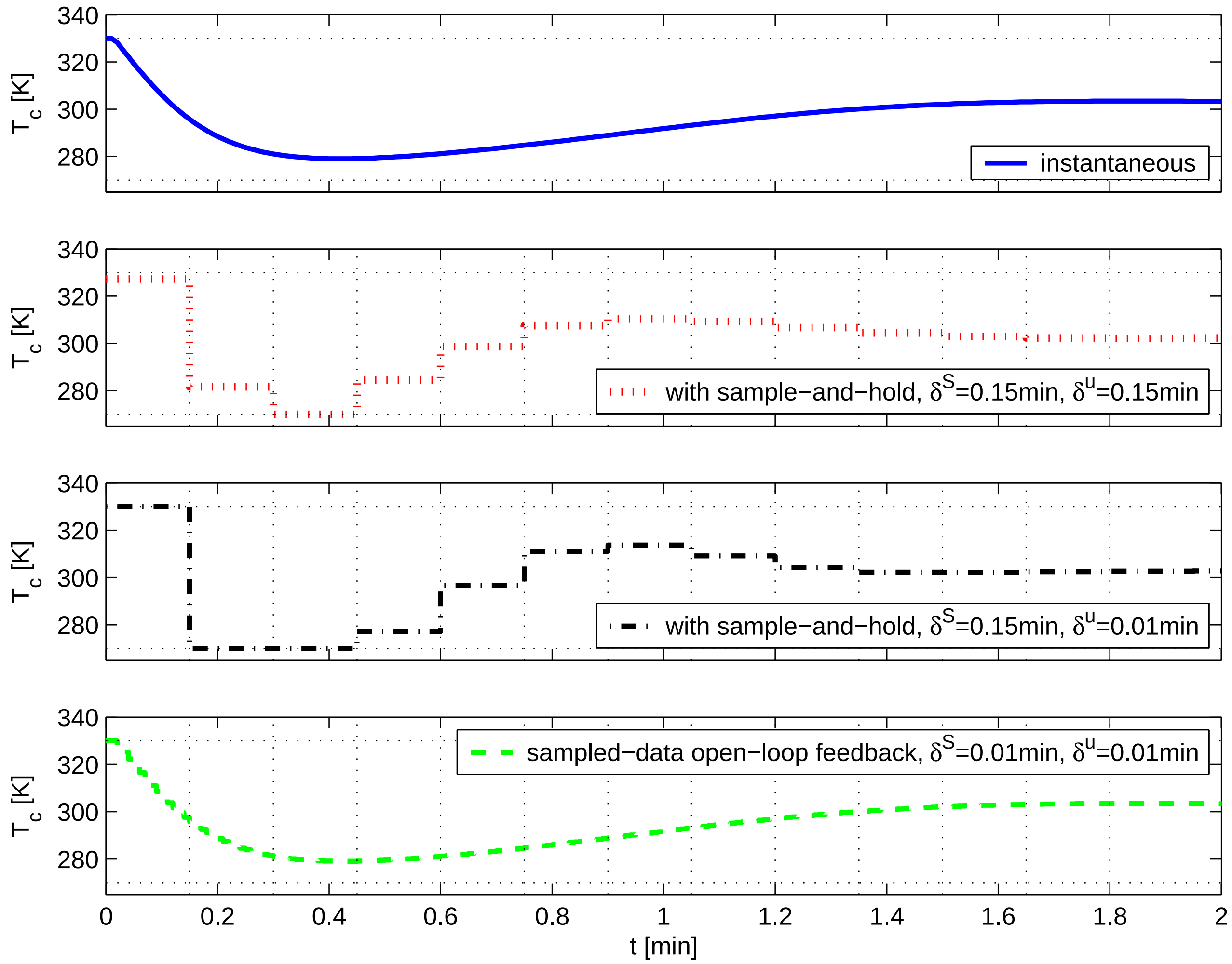


Figure 4.8: Manipulated variable coolant temperature T_c for instantaneous, sample-and-hold, and sampled-data open-loop feedback implementations.

recalculation time δ^r, the performance decreases even further. This can be seen from the dotted curve, which corresponds to a sample-and-hold implementation with $\delta^S = 0.15$min, while the input for the direct solution method is discretized with $\delta^u = 0.01$min.

As for the input-output linearization based sampled-data open-loop feedback implementation, the results underpin some of the advantages that result from an application of an open-loop input in between sampling instants.

4.5 Consideration of Delays

So far, a rather ideal setup was considered. In this section we consider the problem of computational and measurement delays present in the closed-loop. As it will be shown, a sampled-data open-loop feedback allows in a straightforward way to consider such measurement and computational delays by feedforward prediction.

Measurement and computational delays might be present for several reasons; we only mention a few examples here:

- The state information might be delayed, if a computationally involved state estimator, like moving horizon state estimation, is used.
- The state information might be delayed due to off-site measurement of a chemical composition, slow sensors, or a detailed laboratory analysis of a drawn sample.
- The state estimate might be based on computationally intense digital signal processing such as digital image processing. One example might be vision based tracking information of moving objects.
- Possibly present communication delays due to the use of a process control bus for communication between the sensors and the process control system.
- The real-time scheduling present in modern process control systems might lead to a delayed calculation of the sampled-data open-loop feedback due to processes with a higher priority, such as safety related operations.
- The calculation of the sampled-data open-loop feedback itself might be computationally intense and lead to computational delays:
  - One specific example is the often computationally intense solution of the optimal control problem in sampled-data open-loop NMPC. Even so improvements in dynamic optimization have led to efficient numerical solution methods for the open-loop optimal control problem, see Chapter 3, for fast or large scale systems the required solution time is often non negligible.
  - Another example might be the calculation of the open-loop feedback based on the feed-forward simulation of the closed-loop, which also requires a certain, sometimes non-negligible computation time.

In principle it is possible to neglect sufficiently small delays and consider them as small, unknown disturbances. As shown in Chapter 5 sampled-data open-loop feedbacks can reject small disturbances, provided that certain continuity assumptions hold. However, in general it is difficult to estimate the degree of robustness a priory and it might be even possible that the closed-loop does not possess any inherent robustness. Thus, known delays should be always considered by the control, because otherwise the performance might degrade significantly or even instability of the closed-loop can occur. As motivating example consider the CSTR process, as presented Section 4.4.3. Figure 4.10 shows simulation results for the CSTR reactor, neglecting a computational delay, which is assumed to be equal to the recalculation time of 0.15min. As can be seen from the corresponding uncorrected implementation (dashed-doted line), the performance degrades significantly and the delay even leads to instability.

In the following we outline how one can take computational and measurement delays straightforwardly into account, leading to closed-loop stability and a certain degree of recovery of performance. Basically, measurement and computational delays have the same influence on the closed-loop, namely that the obtained input trajectory does not fit anymore to the currently present state. The consideration of the delay is simply based on a sufficiently long feedforward prediction.

4.5.1 Measurement Delays

For simplicity assume that the measurement delay is constant[2]. We denote the delay by δ^c and assume that $\delta^c \leq \underline{\pi}$. Mathematically, a measurement delay can be represented by the fact that at the recalculation time t_i not the true system state $x(t_i)$ is available for feedback, rather only the state at time $t_i - \delta^c$, i.e. $x(t_i - \delta^c)$, is available. Thus, obtaining a consistent initial guess for the sampled-data open-loop feedback calculation is rather simple. It is only necessary to feedforward simulate the system from $x(t_i - \delta^c)$ using the known open-loop input signal $u_{SD}(\cdot; x(t_{i-1}), t_{i-1})$, i.e. $x(t_i)$ is given by $x(t_i) = \hat{x}(t_i)$, where $\hat{x}(t_i)$ is given by the solution of:

$$\dot{\hat{x}}(\tau) = f(\hat{x}(\tau), u_{SD}(\tau; x(t_{i-1}), t_{i-1})), \quad \hat{x}(t_i - \delta^c) = x(t_i - \delta^c), \quad \tau \in [t_i - \delta^c, t_i]. \tag{4.47}$$

Note that the derived stability results are also valid for the case of measurement delays that are considered as proposed. This is immediately clear, since $\hat{x}(t_i) = x(t_i)$. The simulation based consideration of measurement delay thus provides a simple mean to counteract measurement delays that are present in all practical applications.

4.5.2 Computational Delays

For the case of computational delays we assume that at least an upper bound on the computational delay of the form $\bar{\delta}^c \leq \underline{\pi}$ is known. Thus, in the worst case the open-loop feedback trajectory for the recalculation time t_i is only present at time $t_i + \bar{\delta}^c$. We simply propose to overcome this problem by shifting the applied input by the maximum measurement delay. The sampled-data open-loop input is actually calculated for the predicted state at time $t_i + \bar{\delta}^c$ and is applied to the system over the interval $[t_i + \bar{\delta}^c, t_{i+1} + \bar{\delta}^c]$. To achieve this, similarly to the measurement delay case, the state at time $x(t_i)$ is feedforward predicted using the system model to obtain an estimate for the state at time $x(t_i + \bar{\delta}^x)$. The input applied to the system is thus actually given by:

$$u(t) \;\; = \;\; u_{SD}(t; \hat{x}(t_i + \bar{\delta}^c), t_i + \bar{\delta}^c) \tag{4.48}$$

where t_i is the previous closest recalculation instant to the time t, and where $\hat{x}(t_i + \bar{\delta}^c)$ is given by:

$$\hat{x}(\tau) \; = \; f(\hat{x}(\tau), u_{SD}(\tau; x(t_{i-1} + \bar{\delta}^c), t_{i-1} + \bar{\delta}^c)), \quad \hat{x}(t_i) \; = \; x(t_i), \quad \tau \; \in \; [t_i, t_i + \bar{\delta}^c]. \tag{4.49}$$

This strategy is depicted in Figure 4.9. Similarly to the measurement delay case it is clear that the

[2]Note that the delay can also vary in length, as long as it is known and not longer than $\underline{\pi}$

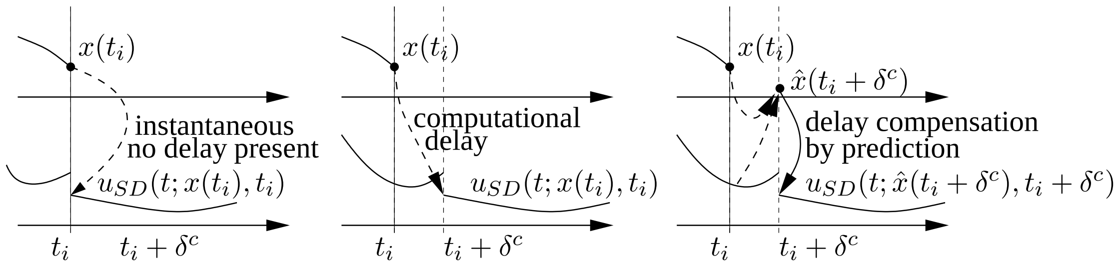


Figure 4.9: Ideal/instantaneous feedback (left), computational delay present (middle), and delay considered (right).

derived stability results do also hold in the delayed case if the delay is considered as proposed. In the proof of Theorem 4.1, for example, one only has to consider a shifted problem starting at time $t_i + \delta^c$, keeping in mind that in the nominal case $\hat{x}(t_i + \delta^c) = x(t_i + \delta^x)$.

The presented approach is a generalization of the results for sampled-data NMPC as presented in (Chen et al., 2000) and (Findeisen and Allgöwer, 2004a) to the sampled-data feedback case.

The derived methods provide simple means for the consideration of measurement and computation delays without loss of stability and a certain degree of recovery of performance in the nominal case. Computational and implementation wise, the approaches only require an additional feedforward simulation of the measured system state using the nominal system model. In the next section we outline the application of the outlined approach to the CSTR example process.

4.5.3 Simulation Example

In this section we apply the outlined computational delay consideration technique to the CSTR example under sampled-data open-loop NMPC. As before we assume that the recalculation instants are equally apart, i.e. $t_i = i\delta^r$, with $\delta^r = 0.15\text{min}$, and that the maximum required solution time $\bar{\delta}^c$ coincides with the recalculation time , i.e. $\bar{\delta}^c = \delta^r$. The prediction horizon is set to $T_p = 1.5$min, and the optimal control problem (4.22) is solved via a direct solution method in Matlab, where the input is parameterized as before as piecewise constant with a discretization time of $\delta^u = 0.01$min. This corresponds to a total number of 150 decisions in the resulting optimization problem, solving the underlying differential equations. As before we want to stabilize the steady-state $T_s = 375K$, $c_{As} = 0.159$mol/L. Figure 4.10 shows the simulation result for the initial condition $c_A(0) = 0.5\text{mol/L}$ and $T(0) = 350$K for different NMPC controller implementations. The solid line shows the behavior in the ideal case, i.e. assuming that no computational delay is present and that the open-loop input can be instantaneously obtained from the state information $x(t_i)$ at the recalculation instants t_i. The dashed-dotted line shows the significantly degraded performance resulting from the sampled-data open-loop input signal implemented with a 0.15min delay. Note that even if the computational delay is reduced to significantly smaller values in the order of 0.03min, instability in form of a yet smaller, but still existing oscillation occurs. The oscillations are even more visible in the manipulated variable, the coolant stream temperature T_c, which for the delay of 0.15min oscillates close to the minimum

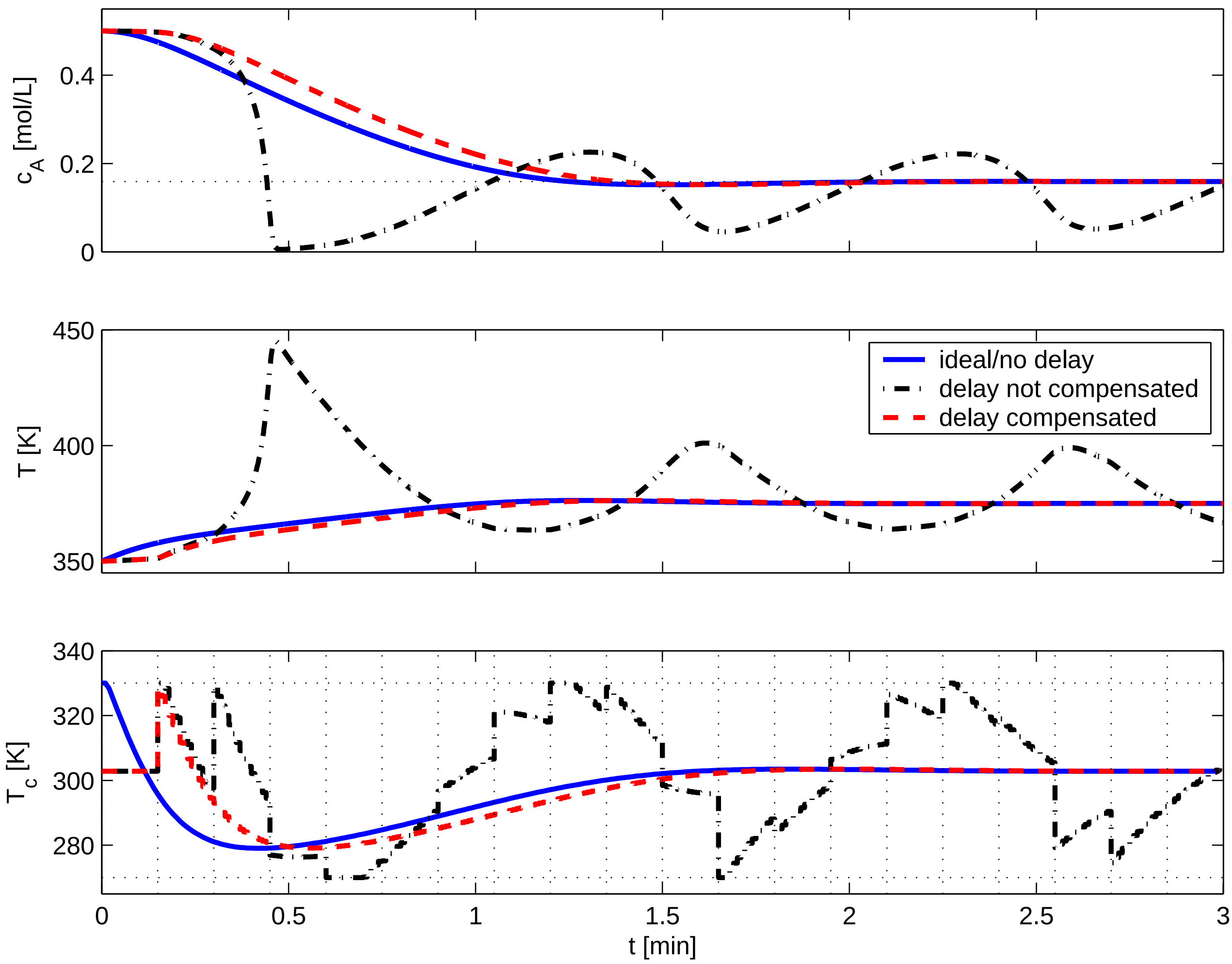


Figure 4.10: Resulting states considering an ideal NMPC controller, an NMPC controller that does neglect the delay, and an NMPC controller that accounts for the delay.

and maximum allowed input values. The dashed line shows the simulation results applying the outlined method. As one would expect, the performance decreases, mainly due to the first control move, which is fixed to the expected steady-state value since no admissible solution of the optimal control problem is yet available. Overall, the performance is satisfying, especially the desired steady-state is nicely reached and no oscillations, neither in the controlled, nor in the manipulated variables are visible. Summarizing, the example clearly underpins the necessity to explicitly account computational and measurement delays. The derived approaches, which are rather simple to implement, allow to account for the delays without loss stability in the nominal undisturbed case.

4.6 Summary and Discussion

In this chapter consider the question of stabilization using sampled-data open-loop feedback. After a short review on the difference between sampled-data open-loop feedback and feedback implementations based on sample-and-hold, we derived new stability conditions for sampled-data open-loop state-feedback. These results for example allow to consider controls that are discontinuous in the state. This is of special interest for predictive control, since it often cannot be guaranteed a priori that the feedback resulting from the solution of the corresponding optimal control problem is continuous in the state.

While the derived conditions seem to be, on a first view, rather restrictive, we outlined the usefulness of these considering two specific sampled-data control controllers: the implementation of sampled-data open-loop feedbacks based on the feedforward simulation of a continuous instantaneous state-feedback, and sampled-data open-loop feedback based on a rather general NMPC setup. The open-loop sampled-data feedback strategies where exemplified considering the control of a simple CSTR. Even so that the considered example is rather simple it clearly underpins the advantages of sampled-data open-loop feedback in comparison to an implementation via sample-and-hold. In the nominal case, i.e. in the absence of external disturbances and model-plant mismatch, it is even possible to recover the performance of an instantaneous feedback.

Even so that sampled-data open-loop feedback allows coping with rarely available state measurements and varying recalculation times, it is, as exemplified in the CSTR example, inherently sensible with respect to delays. To overcome this sensitivity we proposed two techniques for delay consideration using feedforward simulation. While in the case of measurement delays it is necessary that the delay is exactly known, which is often the case, for computational delays it is only necessary to know a maximum upper bound. The outlined techniques allow a simple, easily implementable technique to account for such delays.

However, it is important to note that the derived results are all based on the consideration of an idealized, nominal setup. Especially, it is assumed that neither model-plant mismatch, nor external disturbances are present. This condition is certainly not valid for practical applications and thus makes the examination of the influence of external disturbances and model-plant mismatch on the performance and stability indispensable. The following chapter considers this question from an analysis point of view. Specifically it is analyzed, under which conditions sampled-data open-loop feedbacks possess inherent robustness properties, at least with respect to small disturbances.

Chapter 5

Inherent Robustness Properties of Sampled-data Open-loop Feedbacks

The previous chapter focused on nominal stability results for sampled-data open-loop feedback, including sampled-data open-loop NMPC. In reality, however, model-plant mismatch, exogenous disturbances, unknown delays, numerical errors, and state estimation errors are present. Analyzing the influence of such unknown disturbances is especially important in the case of sampled-data open-loop feedback, since the state information is only fed back at the recalculation times, i.e. the controller cannot immediately react to disturbances. Is it still possible to achieve stability and good performance, at least in the case of small disturbances? Also, what type of performance and stability can be expected, if the disturbances are persistent? In this chapter we try to provide some answers to these questions.

We do not consider the design of robustly stabilizing controllers. Rather we analyze the inherent robustness properties of sampled-data open-loop feedback. Especially, we show that sampled-data open-loop feedback possesses inherent robustness properties if the decreasing function is locally Lipschitz. The results are of practical interest as they underpin that small disturbances, for example due to model-plant mismatch or numerical errors, can be tolerated.

The derived results are related to robustness results for discrete time systems (Scokaert et al., 1997) as well as to results on sampled-data feedback considering sample-and-hold elements for the input (Kellett and Teel, 2002; Kellett et al., 2002; Kellett, 2002; Clarke et al., 2000; Clarke et al., 1997).

We begin in Section 5.1 with a short motivation, considering the CSTR example of Section 4.4.3. Section 5.2 states the considered setup and Section 5.3 outlines the considered stability notation. In Section 5.4 robustness results with respect to additive disturbances are presented. Section 5.5 presents results with respect to input disturbances, such as numerical errors and neglected delays. The question of robustness with respect to measurement disturbances and state estimation errors is considered in Section 5.6. Figure 5.1 shows a sketch of the considered type of disturbances.

The results derived are based on results for sampled-data open-loop NMPC as presented in (Findeisen et al., 2003e; Findeisen et al., 2003d; Findeisen et al., 2003c; Findeisen et al., 2003b).

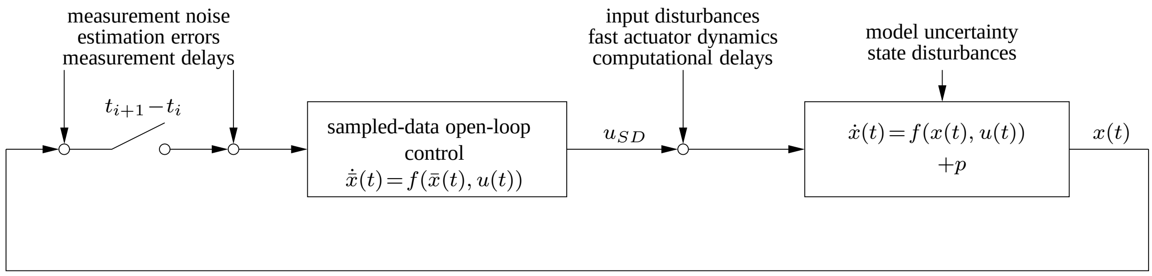


Figure 5.1: Considered robustness setup. The variable p represents an additive model-plant mismatch.

5.1 Motivation

We consider the CSTR example of Section 4.4.3. However, now we assume that a model-plant mismatch is present, i.e. in the real system the cooling temperature is disturbed by a "multiplicative" uncertainty:

$$\dot{c}_{Ar} = \frac{q}{V}(c_{Af} - c_{Ar}) - k_0 e^{\frac{-E}{RT_r}} c_{Ar} \tag{5.1}$$

$$\dot{T}_r = \frac{q}{V}(T_f - T_r) + \frac{-\Delta H}{\rho C_p} k_0 e^{\frac{-E}{RT_r}} c_{Ar} + \frac{UA}{V\rho C_p}(\gamma_{\text{multdist}}(T_c - T_{cs}) + T_{cs} - T_r). \tag{5.2}$$

To distinguish the real system from the model (4.36), we use T_r and c_{Ar} for the real temperature and concentration. The model-plant mismatch stems from the multiplicative term γ_{multdist}. As can be seen, for $\gamma_{\text{multdist}} = 1$ the real plant and the model (4.36) coincide.

Figure 5.2, and Figure 5.3 show simulation results for instantaneous and sampled-data open-loop feedback implementations of NMPC and the I/O linearizing controller outlined in Section 4.4.3 for different values of γ_{multdist}. All shown simulations start at $t = 0$ from the same initial condition $c_A(0) = 0.5\text{mol/L}$ and $T(0) = 350\text{K}$. The parameters for the I/O linearizing controller and the NMPC controller are the same as in Section 4.4.3. For the sampled-data implementations the recalculation time is fixed to $\delta^r = 0.15\text{min}$. For the simulation of the I/O linearizing controller, as shown in Figure 5.2, λ_{iolin} is set to 3. The left plots in Figure 5.2 show the behavior of an instantaneously implemented I/O linearizing controller for γ_{multdist} values of 1.0 (nominal case), 2, 3, 4, 5 and 6. As can be seen from the plot of the reactor temperature (top figure) and the coolant temperature T_c (bottom figure), the controller is able to stabilize the CSTR for all values of γ_{multdist} and achieve nice overall performance. However, this changes for a sampled-data open-loop implementation of the controller, as shown in the right plots of Figure 5.2. Since the mismatch between the model and the plant is only "detected" at the recalculation instants, the performance even degrades for small values of γ_{multdist}. Note that for the sampled-data open-loop controller only the results for γ_{multdist} values between 1.0 and 2.0 are shown. For a value of $\gamma_{\text{multdist}} = 1.8$ strong oscillations occur. Furthermore, due to the model-plant mismatch, the controller is not able to bring the reactor temperature to the corresponding set-point value. Rather the controller is only able to drive the system into a small band around the set-point.

The plots for the NMPC implementations, see Figure 5.3 show similar behavior. The optimal control problem is, as in Section 4.4.3, solved in Matlab, considering for the instantaneous and sampled-data

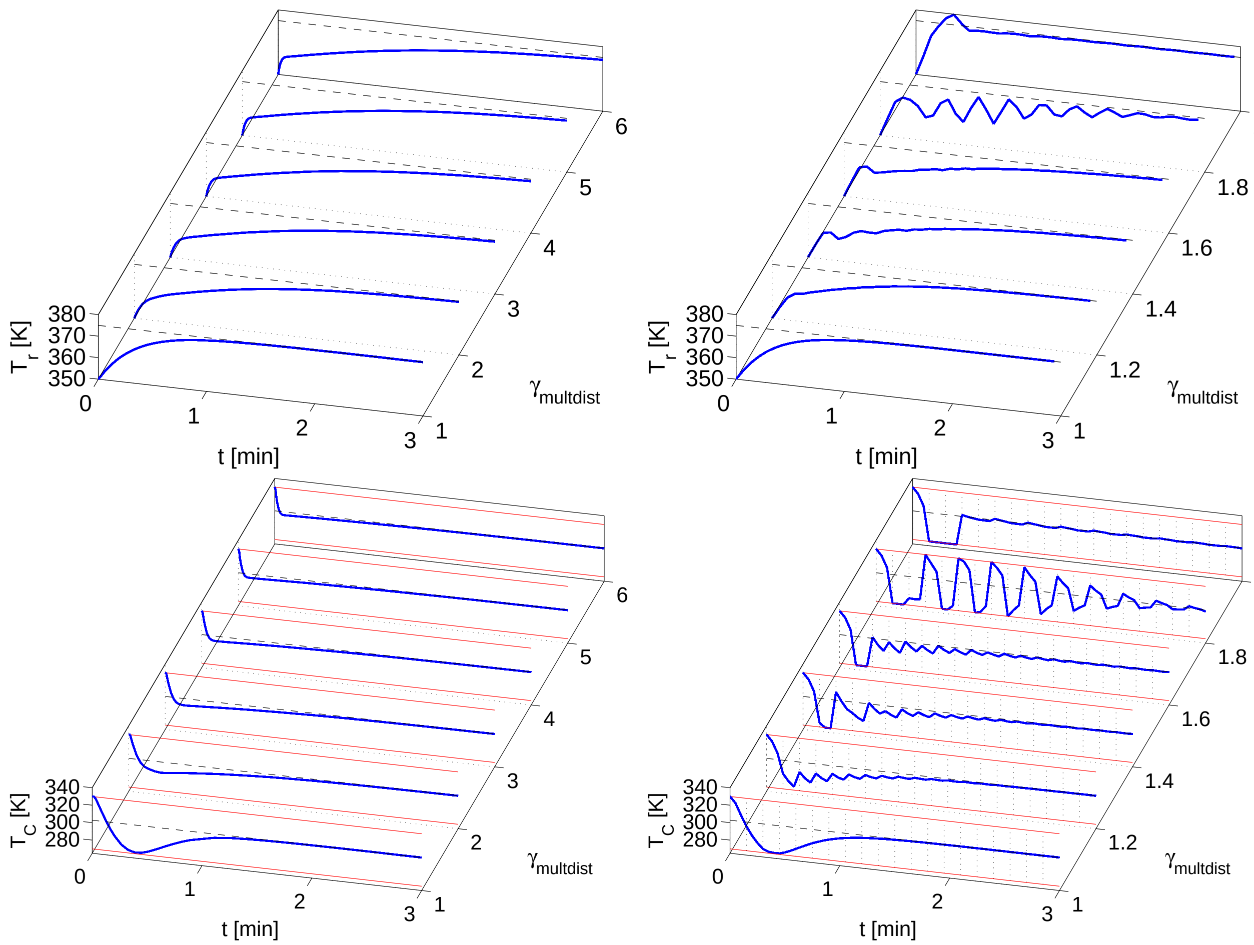


Figure 5.2: Instantaneous I/O linearizing controller (left) and sampled-data open-loop I/O linearizing controller (right) for a recalculation time of $\delta^r = 0.15$min.

implementation an input discretization off $\delta^u = 0.01$min, which is a sufficiently fine input parameterization. The main difference to the I/O linearizing controller is that in the sampled-data case the drastic change in the input signal at the recalculation times is even more visible, leading to a further decreased performance. This can be explained by the fact that the NMPC controller predicts over a rather long horizon into the future, using the incorrect model of the system. Summarizing, the instantaneous controllers are able to nicely counteract the unknown model uncertainty and stabilizes the system. For the sampled-data controllers, however, the performance decreases dramatically with increasing values of $\gamma_{\mathrm{multdist}}$. Nevertheless, both sampled-data controllers are able to stabilizes the system in a practical sense, e.g. keep the input and the state bounded.

Motivated by the simulations we consider in the following the question, under which conditions sampled-data open-loop feedback controllers, specifically sampled-data open-loop NMPC, possess inherent robustness with respect to disturbances? Furthermore, what kind of stability can be achieved, if the disturbance is not vanishing over time?

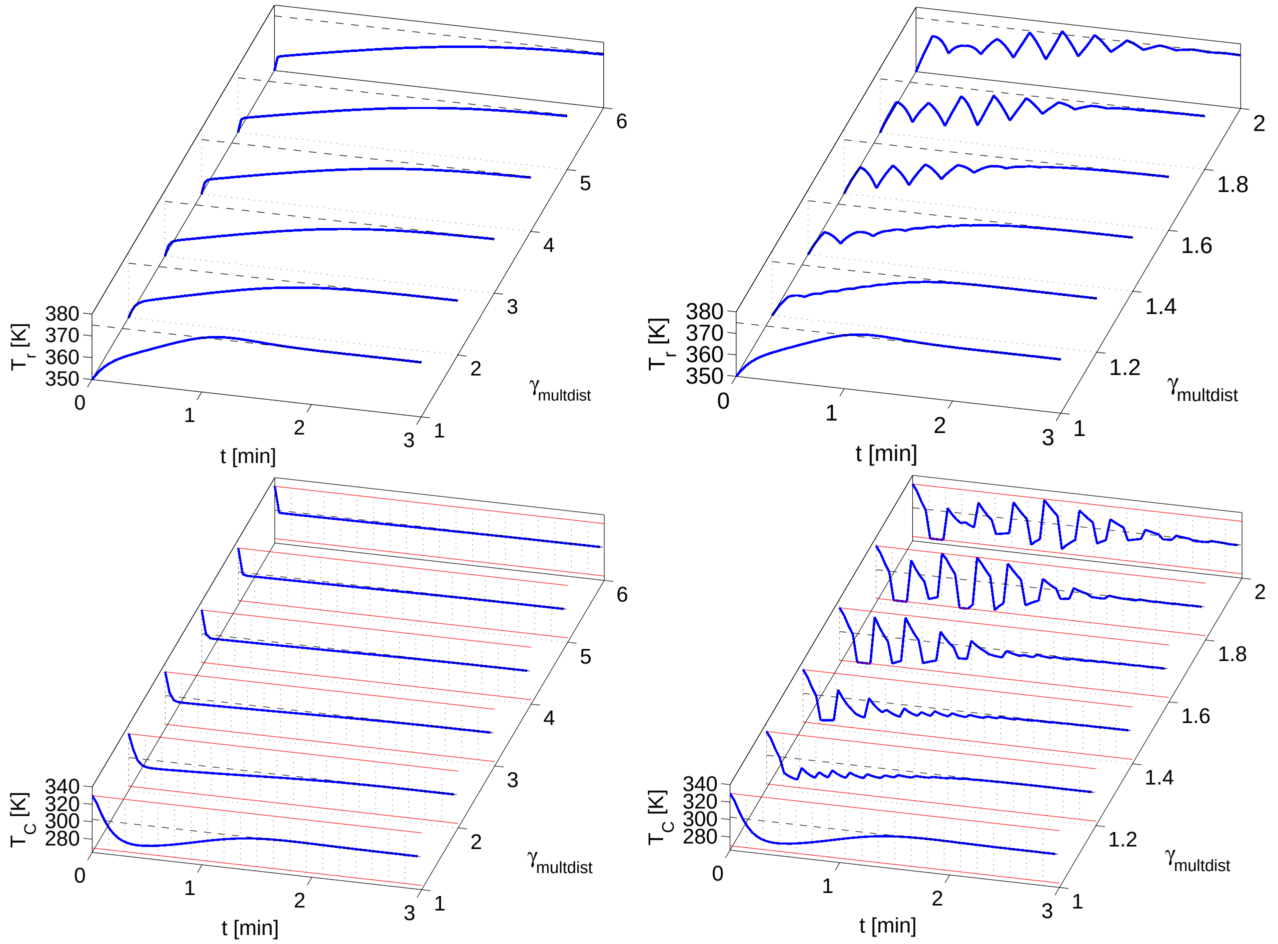


Figure 5.3: Instantaneous NMPC (left) and open-loop sampled-data NMPC (right) for a recalculation time of $\delta^r = 0.15$min.

5.2 Setup

We consider that the nominal system is given by

$$\dot{x}(t) = f(x(t), u(t)), \qquad t \geq 0, \quad x(0) = x_0, \tag{5.3}$$

where $x(t) \in \mathbb{R}^n$ denotes the system state, and $u(t) \in \mathbb{R}^m$ denotes the input. With respect to the vector field f we assume that:

Assumption 5.1
The vector field $f : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}^n$ *is continuous in* u *and locally Lipschitz in* x. *Furthermore,* $f(0,0) = 0$.

Here $\mathcal{X} \subseteq \mathbb{R}^n$ denotes, as before, the set of feasible states and the compact set $\mathcal{U}$ with $\mathcal{U} \subseteq \mathbb{R}^m$ denotes the set of feasible inputs. We assume, that

Assumption 5.2 $(0,0) \in \mathcal{X} \times \mathcal{U}$.

The input to the system (5.3) is given by a sampled-data open-loop feedback controller

$$u(t) = u_{SD}(t; x(t_i), t_i). \tag{5.4}$$

The recalculation instants t_i are defined via a partition π. With respect to the feedback u_{SD} we assume that it stabilizes the origin of the nominal system with a region of attraction $\mathcal{R} \subseteq \mathcal{X}$, $0 \in \mathcal{R}$, and that a Lipschitz assumption on the corresponding decreasing function is satisfied. In the spirit of the nominal stability results of Chapter 4, this is covered by the following assumption:

Assumption 5.3 (Nominal stability of the sampled-data open-loop feedback)

1. *The input generator u_{SD} is admissible with respect to a set $\mathcal{R}$, the input and state constraint sets $\mathcal{U}$, $\mathcal{X}$, and the partition π.*

2. *There exists a locally Lipschitz continuous positive definite function $\alpha : \mathcal{R} \to \mathbb{R}^+$ and a continuous positive definite function $\beta : \mathcal{R} \to \mathbb{R}^+$, such that for all $t_i \in \pi$, $x(t_i) \in \mathcal{R}$ and $\tau \in [0, t_{i+1} - t_i)$*

   (a)
   $$\alpha(x(t_i + \tau; x(t_i), u_{SD}(\cdot; x(t_i), t_i))) - \alpha(x(t_i)) \leq -\int_{t_i}^{t_i+\tau} \beta(x(s; x(t_i), u_{SD}(\cdot; x(t_i), t_i)))ds \quad (5.5)$$

   holds.

   (b) for all compact strict subsets $\mathcal{S} \subset \mathcal{R}$ there is at least one level set $\Omega_c = \{x \in \mathcal{R} | \alpha(x) \leq c\}$ s.t. $\mathcal{S} \subset \Omega_c$.

Remark 5.1 *A direct consequence of the local Lipschitz continuity assumption on $\alpha(x)$ in $\mathcal{R}$ is that for any level set $\Omega_c \subseteq \mathcal{R}$ there exists a Lipschitz constant L_α such that for any $x_1, x_2 \in \Omega_c$:*

$$\|\alpha(x_1) - \alpha(x_2)\| \leq L_\alpha \|x_1 - x_2\|. \quad (5.6)$$

We denote α in the following as decreasing or Lyapunov like function. Assumption 5.3 guarantees that all conditions of Theorem 4.1, with respect to the set $\mathcal{A} = \{0\}$, hold, i.e. the sampled-data open-loop feedback u_{SD} stabilizes the origin with a region of attraction that at least contains $\mathcal{R}$. The Lipschitz assumption on the decreasing function α and the existence of compact level sets Ω_c is necessary to achieve the desired robustness properties. This requirement is in correspondence with recent results on the stability and robustness of discontinuous feedbacks with sample-and-hold (Kellett et al., 2002; Kellett, 2002).

Assumption 5.3 is in general satisfied for sampled-data open-loop feedbacks derived from locally Lipschitz continuous instantaneous feedback laws as outlined in Section 4.4.1. While stabilizing open-loop sampled-data NMPC schemes automatically satisfy the decrease condition (5.5), the satisfaction of the uniform continuity assumption on α and the existence of compact level sets Ω_c is typically not ensured. This is a direct consequence of the fact that NMPC controllers can stabilize systems that require a discontinuous input, and thus might lead to discontinuity in the decreasing function (Fontes, 2003; Fontes, 2000a; Meadows et al., 1995; Grimm et al., 2004a). This issue is further discussed in Section 5.7.

5.3 Considered Type of Stability

We consider persistent disturbances and the repeated application of open-loop inputs, i.e. we cannot react instantaneously to disturbances. Thus, asymptotic stability cannot be achieved, and the nominal region of attraction $\mathcal{R}$ can in general not be rendered invariant under disturbances. As a consequence, we desire only "ultimate boundedness" results, i.e. we desire that the norm of the state after some time becomes small. Furthermore, we show that the bound can be made arbitrarily small depending on the bound on the disturbance and the sampling time (*practical stability*), and that the region where this holds can be made an arbitrarily inner approximation with respect to $\mathcal{R}$ (*semi-regional*). In view of Assumption 5.3 and for simplicity of presentation, we parameterize these regions with level sets.

Specifically, we derive bounds for the maximum allowable disturbance and sampling time that ensure that the state converges from any arbitrary level set of initial conditions $\Omega_{c_0} \subset \mathcal{R}$ in finite time to an arbitrary small set Ω_γ around the origin without leaving a desired set $\Omega_c \subset \mathcal{R}$, compare Figure 5.4. Certainly, the maximum allowable disturbance depends on the size of the region of convergence Ω_γ

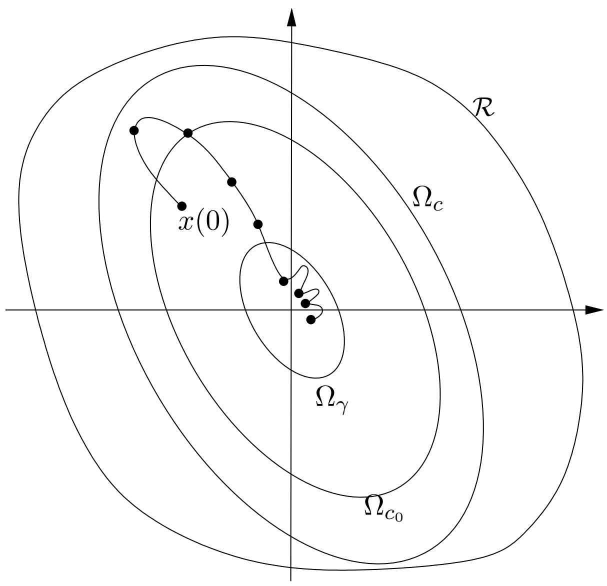


Figure 5.4: Set of initial conditions Ω_{c_0}, maximum attainable set Ω_c, desired region of convergence Ω_γ and nominal region of attraction $\mathcal{R}$.

and on the "distance" between Ω_c and Ω_{c_0}.

The derived results are based on the observation that small disturbances and model uncertainties lead to a (small) difference between the nominal open-loop state and the real state. As will be shown, the influence of the disturbance on the decreasing function α can be bounded by

$$\alpha(x(t_{i+1})) - \alpha(x(t_i)) \leq - \int_{t_i}^{t_{i+1}} \beta(x(\tau; x(t_i), u_{SD}(\cdot; x(t_i), t_i)))d\tau + \epsilon(t_i), \tag{5.7}$$

where ϵ corresponds to the "disturbance contribution". Thus, if the disturbance contribution ϵ "scales" with the size of the disturbance (it certainly also scales with the recalculation time $t_{i+1} - t_i$), one can achieve contraction of the level sets, at least at the recalculation instants. Since the integral contribution in (5.7) depends on the distance of the system state to the origin, while the disturbance contribution mainly depends on the size of the disturbances, the decrease cannot continue until reaching the origin, i.e. in general only practical stability can be achieved.

For the robustness derivations we need the function $\Delta\alpha_{\min}(c,\gamma)$ which is defined as:

Definition 5.1 ($\Delta\alpha_{\min}$)
For any $c>\gamma>0$ with $\Omega_c \subset \mathcal{R}$, the value of $\Delta\alpha_{\min}(c,\gamma)$ is defined as

$$\Delta\alpha_{\min}(c,\gamma) = \min_{\substack{x_0\in\Omega_c/\Omega_\gamma \\ t_i\in\pi}} \int_{t_i}^{t_{i+1}} \beta(\bar{x}(s;x_0,u_{SD}(\cdot;x_0,t_i)))ds. \tag{5.8}$$

Here $\bar{x}$ is the state of the nominal system under the nominal sampled-data open-loop feedback, i.e.

$$\dot{\bar{x}}(s) = f(\bar{x}(s),u_{SD}(s;x(t_i),t_i)), \qquad s\in[t_i,t_{i+1}],\ \bar{x}(t_i)=x_0. \tag{5.9}$$

Note that for any $c>\gamma>0$ with $\Omega_c\subset\mathcal{R}$, $\Delta\alpha_{\min}(c,\gamma)$ is nontrivial and finite. In general it is difficult to obtain an explicit expression or even a good lower bound for $\Delta\alpha_{\min}$. In the case that the recalculation instants are equidistant, the calculation is simplified, since the second minimization argument, the time t_i, can be removed.

5.4 Robustness to Additive Disturbances

We first examine the robustness with respect to additive disturbances. Specifically, we consider that the disturbances affecting the system lead to the following modified system equation:

$$\dot{x}(t) = f(x(t),u_{SD}(t;x(t_i),t_i)) + p(t). \tag{5.10}$$

All appearing disturbances and model-plant uncertainties are lumped in the disturbance term p. With respect to the additive disturbance p we can derive the following result

Theorem 5.1 (Robustness with respect to additive disturbances)
Given arbitrary level sets $\Omega_\gamma\subset\Omega_{c_0}\subset\Omega_c\subset\mathcal{R}$ and assume that Assumptions 5.1- 5.3 hold. Then, there exists a constant $p_{\max}>0$, such that for any disturbance satisfying for all $t_i\in\pi$

$$\left\|\int_{t_i}^{t_i+\tau} p(s)ds\right\| \le p_{\max}\tau, \quad \tau\in[0,t_{i+1}-t_i], \tag{5.11}$$

the trajectories of the disturbed system for any $x_0\in\Omega_{c_0}$

$$\dot{x}(t) = f(x(t),u_{SD}(t;x(t_i),t_i)) + p(t), \qquad x(0)=x_0, \tag{5.12}$$

exist for all times, will not leave the set Ω_c, $x(t_i)\in\Omega_{c_0}$ $\forall i\ge 0$, and there exists a finite time T_γ such that $x(\tau)\in\Omega_\gamma$ $\forall\tau\ge T_\gamma$.

Remark 5.2 *The bound* (5.11) *ensures existence of solutions and convergence to the set Ω_c. Examples of disturbances satisfying condition* (5.11) *are constant additive disturbances and time varying disturbances. Note that it is not necessary to require that the disturbance vanishes over time, since we do not desire to achieve asymptotic convergence. In general, the disturbances also depends on the state and input or sampling time. The derived result can be used in this case, if the integrability condition* (5.11) *on p holds.*

Proof: The proof consists of 3 parts. In the first part we establish conditions guaranteeing that the state does not leave the set Ω_c for all $x(t_i)\in\Omega_{c_0}$. In the second part we establish conditions such that the states converge in finite time to the set $\Omega_{\gamma/2}$. The last part ensures that for all $x(t_i)\in\Omega_{\gamma/2}$ the state does not leave the set Ω_γ.

First part ($x(t_i+\tau)\in\Omega_c\ \forall x(t_i)\in\Omega_{c0}$):
We start by comparing the nominal (predicted) trajectory $\bar{x}$ and the trajectory of the real state x starting from the same initial state at a $t_i\in\pi$ with $x(t_i)\in\Omega_{c_0}$. One specific t_i is $t=0$, for which we know that $x(0)\in\Omega_{c_0}$. First note that $x(t_i+\tau)$ and $\bar{x}(t_i+\tau)$ can be written as (skipping for sake of notation the additional arguments the state depends on):

$$x(t_i+\tau) \;=\; x(t_i)+\int_{t_i}^{t_i+\tau}\left(f(x(s),u_{SD}(s;x(t_i),t_i))+p(s)\right)ds\;, \tag{5.13}$$

$$\bar{x}(t_i+\tau) \;=\; x(t_i)+\int_{t_i}^{t_i+\tau} f(\bar{x}(s),u_{SD}(s;x(t_i),t_i))ds. \tag{5.14}$$

The existence of solutions for x and $\bar{x}$ is ensured for a sufficiently small τ, if we assume that p is integrable, and bounded, since $\Omega_{c0}\subset\Omega_c\subset\mathcal{R}$. Subtracting $\bar{x}$ from x and applying the triangle inequality, we obtain

$$\|x(t_i+\tau)-\bar{x}(t_i+\tau)\|\le\int_{t_i}^{t_i+\tau}\left(\|f(x(s),u_{SD}(s;x(t_i),t_i))-f(\bar{x}(s),u_{SD}(s;x(t_i),t_i))\|\right)ds + \left\|\int_{t_i}^{t_i+\tau}p(s)ds\right\|. \tag{5.15}$$

Assuming that τ is sufficiently small, such that $x(t+\tau)\in\Omega_c$ we can utilize the Lipschitz property of f in x. Denoting by L_{fx} the corresponding Lipschitz constant of f we obtain:

$$\|x(t_i+\tau)-\bar{x}(t_i+\tau)\|\le\int_{t_i}^{t_i+\tau}\left(L_{fx}\|x(s)-\bar{x}(s)\|\right)ds+\left\|\int_{t_i}^{t_i+\tau}p(s)ds\right\|, \tag{5.16}$$

which leads to

$$\|x(t_i+\tau)-\bar{x}(t_i+\tau)\|\le\int_{t_i}^{t_i+\tau}L_{fx}\|x(s)-\bar{x}(s)\|\,ds+p_{\max}\tau. \tag{5.17}$$

Applying the Gronwall-Bellman inequality we obtain:

$$\|x(t_i+\tau)-\bar{x}(t_i+\tau)\|\le\frac{p_{\max}}{L_{fx}}\left(e^{L_{fx}\tau}-1\right). \tag{5.18}$$

Furthermore, since u_{SD} satisfies Assumption 5.3 we know that

$$\alpha(x(t_i+\tau))-\alpha(x(t_i)) \;\leq\; \alpha(x(t_i+\tau))-\alpha(\bar{x}(t_i+\tau)) \quad (5.19)$$
$$\leq\; L_\alpha \left\|x(t_i+\tau)-\bar{x}(t_i+\tau)\right\|. \quad (5.20)$$

Here we used that $\alpha(\bar{x}(t_i+\tau))-\alpha(x(t_i))\leq 0$, see (5.5). Combining this with (5.18) we obtain

$$\alpha(x(t_i+\tau))-\alpha(x(t_i))\leq \frac{L_\alpha p_{\max}}{L_{fx}}\left(e^{L_{fx}\tau}-1\right). \quad (5.21)$$

Thus, it is immediately clear that if $p_{\max}$ satisfies

$$p_{\max}\leq \frac{L_{fx}}{L_\alpha\left(e^{L_{fx}\bar{\pi}}-1\right)}(c-c_0), \quad (5.22)$$

then $x(t_i+\tau)\in\Omega_c$, $\forall\,\tau\in[0,t_{i+1}-t_i]$ if $x(t_i)\in\Omega_{c_0}$.

Second part ($x(t_i)\in\Omega_{c_0}$ and finite time convergence to $\Omega_{\gamma/2}$):
Assume that (5.22) holds. This assures that $x(t_i+\tau)\in\Omega_c$, $\forall\tau\in[0,t_{i+1}-t_i]$ as long as $x(t_i)\in\Omega_{c_0}$. Assuming that $x(t_i)\notin\Omega_{\gamma/2}\subset\Omega_\gamma\subset\Omega_c$ we know that

$$\alpha(x(t_{i+1}))-\alpha(x(t_i))=\alpha(x(t_{i+1}))-\alpha(\bar{x}(t_{i+1}))+\alpha(\bar{x}(t_{i+1}))-\alpha(x(t_i)). \quad (5.23)$$

Bounding the last two terms on the right via $\Delta\alpha_{\min}$ and the first two terms via (5.18), we obtain

$$\alpha(x(t_{i+1}))-\alpha(x(t_i))\leq \frac{p_{\max}L_\alpha}{L_{fx}}\left(e^{L_{fx}\bar{\pi}}-1\right)-\Delta\alpha_{\min}(c,\gamma/2). \quad (5.24)$$

To achieve convergence to the set $\Omega_{\gamma/2}$ in finite time, we need that the right hand side is strictly less than zero. If we require that

$$p_{\max}\leq \frac{L_{fx}}{L_\alpha\left(e^{L_{fx}\bar{\pi}}-1\right)}\Delta\alpha_{\min}(c,\gamma/4),$$

this is achieved, since then

$$\alpha(x(t_{i+1}))-\alpha(x(t_i))\leq \underbrace{\left(-\Delta\alpha_{\min}(c,\gamma/2)+\Delta\alpha_{\min}(c,\gamma/4)\right)}_{=k_{\text{dec}}}<0. \quad (5.25)$$

Thus, for any $x(t_i)\in\Omega_{c_0}$ we have finite time convergence to the set $\Omega_{\gamma/2}$ for a sampling time t_m that satisfies $t_m-t_i\leq T_\gamma=\frac{c-\gamma/2}{k_{\text{dec}}}$. From this it furthermore follows that $x(t_{i+1})\in\Omega_{c_0}$ for all $x(t_i)\in\Omega_{c_0}$.

Third part ($x(t_{i+1})\in\Omega_\gamma\ \forall x(t_i)\in\Omega_{\gamma/2}$):
This is trivially satisfied following the arguments in the first part of the proof, requiring that

$$p_{\max}\leq \frac{L_{fx}}{L_\alpha\left(e^{L_{fx}\bar{\pi}}-1\right)}\gamma/2. \quad (5.26)$$

Combining the requirements of all three parts leads to an explicit bound for $p_{\max}$:

$$p_{\max}=\frac{L_{fx}}{L_\alpha\left(e^{L_{fx}\bar{\pi}}-1\right)}\min\left\{c-c_0,\Delta\alpha_{\min}(c,\gamma/4),\gamma/2\right\}. \quad (5.27)$$

■

Remark 5.3 *The values $\gamma/4$ and $\gamma/2$ are chosen for simplicity. In principle the corresponding level sets only have to be strict subsets of each other and of Ω_γ.*

Theorem 5.1 establishes robustness of sampled-data open-loop feedbacks with respect to small additive disturbances. The degree of robustness strongly depends on the dynamics of the system, the Lipschitz condition on the decreasing function α, and on the minimum and maximum recalculation time δ_i, i.e. $\bar{\pi}$ and $\underline{\pi}$.

Remark 5.4 *Calculating the robustness bound $p_{\max}$ is difficult, since it is necessary to at least know a lower bound on the minimum decrease $\Delta\alpha_{\min}(c, \gamma/4)$. This requires, that the decreasing function α is known, and that the integral and maximization appearing in* (5.8) *can be performed. Furthermore, it is in general difficult to obtain suitable bounds on the appearing Lipschitz constants. Nevertheless, the result is of value, since it underpins that small additive disturbances can be tolerated.*

5.5 Robustness to Input Disturbances

The derived results can be easily tailored to disturbances that directly act on the input. The consideration of disturbances acting directly on u is of interest, since this covers a series of practically important disturbances such as small computational delays, external influences acting on the input, unconsidered fast actuator dynamics, and numerical errors such as approximated solutions of the optimal control problem in NMPC.

To achieve the results it is necessary to assume that f is locally Lipschitz in u over a compact set $\tilde{\mathcal{U}}$ which is slightly larger then $\mathcal{U}$ with $\mathcal{U} \subset \tilde{\mathcal{U}}$, since the nominal controller could use values on the boundary of $\mathcal{U}$:

Assumption 5.4
The vector field $f : \mathcal{X} \times \tilde{\mathcal{U}} \to \mathbb{R}^n$ is locally Lipschitz in x and u. Furthermore, $f(0,0) = 0$.

We assume that the disturbed input is given by $u_{SD}(t; x(t_i), t_i) + v(t)$. Following the ideas in the first part of the proof of Theorem 5.1, assuming that

$$\left\| \int_{t_i}^{t_i+\tau} v(s)ds \right\| \le v_{\max}\tau, \quad \forall t_i \in \pi,\ \tau \in [t_{i+1} - t_i], \tag{5.28}$$

where $v_{\max}$ is sufficiently small, and that $(u_{SD}(t; x(t_i), t_i) + v(t)) \in \tilde{\mathcal{U}}$, which can always be ensured if $\|v\| \le v_{\text{dist}\mathcal{U},\tilde{\mathcal{U}}} = \min_{u \in \mathcal{U}, v \in \partial\tilde{\mathcal{U}}} \|v - u\|$, where $\partial\tilde{\mathcal{U}}$ denotes the border of $\tilde{\mathcal{U}}$, similarly to equation (5.16) one can obtain an estimate of the influence of the disturbed input on the state:

$$\|x(t_i + \tau) - \bar{x}(t_i + \tau)\| \le \int_{t_i}^{t_i+\tau} L_{fx}\|x(s) - \bar{x}(s)\|ds + L_{fu}v_{\max}\tau. \tag{5.29}$$

Here L_{fu} is the Lipschitz constant of $f(x, u)$ with respect to u over $\Omega_c \times \tilde{\mathcal{U}}$. Applying the Gronwall-Bellman inequality leads to (5.18) with $p_{\max}$ exchanged by $L_{fu}v_{\max}$. The remainder of the proof stays unchanged, thus we obtain the following result for input disturbances:

Theorem 5.2 (Robustness with respect to input disturbances)
Given arbitrary level sets $\Omega_\gamma \subset \Omega_{c_0} \subset \Omega_c \subset \mathcal{R}$ and assume that Assumptions 5.2- 5.4 hold. Then, there exists a constant $v_{\max} > 0$ such that for any disturbance satisfying for all $t_i \in \pi$

$$\left\| \int_{t_i}^{t_i+\tau} v(s)ds \right\| \leq v_{\max}\tau, \qquad \tau \in [0, t_{i+1} - t_i], \tag{5.30}$$

and

$$\|v(t)\| \leq v_{\text{dist}\mathcal{U},\tilde{\mathcal{U}}}, \quad t \geq 0, \tag{5.31}$$

the trajectories of the disturbed system for any $x_0 \in \Omega_{c_0}$

$$\dot{x}(t) = f(x(t), u_{SD}(t; x(t_i), t_i) + v(t)), \qquad x(0) = x_0, \tag{5.32}$$

exist for all times, will not leave the set Ω_c, $x(t_i) \in \Omega_{c_0}$ $\forall i \geq 0$, and there exists a finite time T_γ such that $x(\tau) \in \Omega_\gamma$ $\forall \tau \geq T_\gamma$.

Proof: The proof is nearly equivalent to the proof of Theorem 5.1 and thus omitted here. ■

A bound for $v_{\max}$ similarly to the additive state disturbance case is given by:

$$v_{\max} = \frac{L_{fx}}{L_\alpha L_{fu} \left(e^{L_{fx}\bar{\pi}} - 1\right)} \min \left\{c - c_0, \Delta\alpha_{\min}(c, \gamma/4), \gamma/2\right\}, \tag{5.33}$$

where L_{fx} and L_{fu} are the Lipschitz constants of f with respect to x and u over $\Omega_c \times \tilde{\mathcal{U}}$.

Besides the important case of disturbances that directly act on the input, the derived result has a series of direct implications.

Numerical approximation errors:
One direct implication of this result is that approximated solutions to the optimal control problem in NMPC can be tolerated, if the approximation error is sufficiently small. Such approximated solutions can for example result from the numerical integration of the differential equations, or errors due to the application of direct solution approaches where the input is parameterized. Furthermore, Theorem 5.2 gives a theoretical foundation for the real-time iteration scheme as outlined in Section 3.5, in which only one Newton step optimization is performed per sampling instant (Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2002; Diehl, Findeisen, Schwarzkopf, Uslu, Allgöwer, Bock and Schlöder, 2003). Based on similar ideas, we have shown in (Diehl et al., 2004; Diehl, Findeisen, Allgöwer, Schlöder and Bock, 2003) that the real-time iteration scheme for discrete time systems does lead to nominal stability under certain conditions.

Computational delays:
The derived result underlines that sufficiently small computational delays can be tolerated. Since the state on which the input calculation is based on remains unchanged, it becomes immediately clear that

condition (5.28) is satisfied if the delay is sufficiently small. In this case condition (5.31) vanishes, since the resulting input is only shifted in time. This result is of special interest for open-loop sampled-data NMPC, since delays will always be present even for fast calculations. It underlines, that if the delay is sufficiently small, the closed-loop achieves practical stability. In case that the delay is known and rather large it is, nevertheless, necessary to employ delay compensation techniques as outlined in Section 4.5.

Neglected fast actuator dynamics:
One further application of the derived result might be the question, if in the case of neglected, but fast actuator dynamics, practical stability can be guaranteed. In principle this is possible, following ideas presented in (Kellett et al., 2002) for the case of sampled-data feedback with sample-and-hold elements. We do not go into details here, since the derivation strongly depends on the actuator dynamics, and since this requires a series of rather technical assumptions to hold. However, we note that in principle fast neglected actuator dynamics can be tolerated, if the speed of the actuator is sufficiently high.

5.6 Robustness to Measurement and State Estimation Errors

In this section we consider the problem of measurement and state estimation errors. The derived result lays the basis for the output-feedback results presented in Chapter 6.

Instead of the real system state $x(t_i)$ we assume that at every sampling instant only a disturbed or estimated state $x(t_i) + e(t_i)$ is available. The disturbance $e(t_i)$ could for example be the result of measurement noise, small measurement delays, or state estimation errors. Instead of the optimal feedback (2.9) the following "disturbed" feedback is applied:

$$u(t;\hat{x}(t_i))=u_{SD}(t;x(t_i)+e(t_i),t_i), \quad t\in[t_i,t_{i+1})\,. \tag{5.34}$$

Note that only the state and disturbance $e(t_i)$ at the recalculation time is of interest for the robustness, the influence of disturbances on the state estimate and measurements in between recalculation times does not influence the achieved results.

Similar considerations as in the additive disturbance case or the input disturbance case lead to the following theorem:

Theorem 5.3 (Robustness with respect to measurement and state estimation disturbances)
Given arbitrary level sets $\Omega_\gamma\subset\Omega_{c_0}\subset\Omega_c\subset\mathcal{R}$ and assume that Assumptions 5.1- 5.3 hold. Then, there exists a constant $e_{\max}>0$ such that for any measurement disturbance and state estimation error $e(t_i)$ satisfying for all $t_i\in\pi$

$$\|e(t_i)\|\leq e_{\max}, \tag{5.35}$$

the trajectories of the system for any $x_0\in\Omega_{c_0}$

$$\dot{x}(t)=f(x(t),u_{SD}(t;x(t_i)+e(t_i),t_i)), \qquad x(0)=x_0, \tag{5.36}$$

exist for all times, will not leave the set Ω_c*,* $x(t_i)\in\Omega_{c_0}\ \forall i \geq 0$*, and there exists a finite time* T_γ *such that* $x(\tau)\in\Omega_\gamma\ \forall \tau \geq T_\gamma$*.*

Proof: The proof is similar to the proof of Theorem 5.1 divided in three parts.

First part ($x(t_i+\tau)\in\Omega_c\ \forall x(t_i)\in\Omega_{c0}$*):*
We consider the difference in the value function between the initial state $x(t_i)\in\Omega_{c_0}$ at a sampling time t_i and the developing state $x(t_i+\tau; x(t_i), u_{SD}(\cdot; x(t_i)+e(t_i), t_i))$. For simplicity of notation and ease of understanding we use the following short notation: $\hat{x}_i = x(t_i) + e(t_i)$, $x_i = x(t_i)$, and $e_i = e(t_i)$. Furthermore, $u_{\hat{x}_i}$ denotes the open-loop input resulting from $\hat{x}(t_i)$, i.e. $u_{\hat{x}_i} = u_{SD}(\cdot; x(t_i) + e(t_i), t_i)$. By similar considerations as for Theorem 5.1 we know that at least as long as the state does not leave the set Ω_c the following inequality is valid

$$\begin{aligned}\alpha(x(\tau;x_i,u_{\hat{x}_i}))-\alpha(x_i) = {} & \alpha(x(\tau;x_i,u_{\hat{x}_i})) - \alpha(x(\tau;\hat{x}_i,u_{\hat{x}_i})) \\ & + \alpha(x(\tau;\hat{x}_i,u_{\hat{x}_i})) - \alpha(\hat{x}_i) + \alpha(\hat{x}_i) - \alpha(x_i). \quad (5.37)\end{aligned}$$

One way to ensure that $\hat{x}_i\in\Omega_c$ if $x_i\in\Omega_{c_0}$ is to require that $L_\alpha \leq c - c_0$. Following the derivation of the previous two proofs, the first two terms can then be bounded via L_α and the Gronwall-Bellman inequality by:

$$\alpha(x(t_i+\tau;x_i,u_{\hat{x}_i})) - \alpha(x(t_i+\tau;\hat{x}_i,u_{\hat{x}})) \leq L_\alpha e^{L_{fx}\tau}\|e_i\|. \quad (5.38)$$

Thus, we obtain:

$$\alpha(x(t_i + \tau;x_i,u_{\hat{x}})) - \alpha(x_i) \leq L_\alpha e^{L_{fx}\tau}\|e_i\| - \int_{t_i}^{t_i+\tau}\beta(x(s;\hat{x}_i,u_{\hat{x}_i}))ds + L_\alpha\|e_i\| \quad (5.39)$$

Since the contribution of the integral is always negative, it follows that if

$$e_{\max} \leq \frac{1}{L_\alpha\left(e^{L_{fx}\bar{\pi}}+1\right)}(c-c_0) \quad (5.40)$$

which implies that $L_\alpha e_{\max} \leq c - c_0$, then $x(t_i+\tau)\in\Omega_c\ \forall\tau\in(t_{i+1}-t_i)$.

Second part ($x(t_i)\in\Omega_{c_0}$ *and finite time convergence to* $\Omega_{\gamma/2}$*):*
We assume that (5.40) holds and that $x(t_i)\in\Omega_{c_0}$. This assures that $x(t_i+\tau)\in\Omega_c$, $\forall\tau\in[0, t_{i+1}-t_i]$. Considering (5.39) it is clear, that if $e_{\max}$ satisfies

$$L_\alpha\left(e^{L_{fx}\bar{\pi}}+1\right)e_{\max} \leq \frac{1}{2}\Delta\alpha_{\min}(c,\gamma/4) \quad \text{and} \quad L_\alpha e_{\max} \leq \gamma/4,$$

then we achieve finite time convergence from any $x_i\in\Omega_{c_0}\backslash\Omega_{\gamma/4}$ to the set $\Omega_{\gamma/2}$ for a sampling time t_m satisfying $t_m - t_i \leq T_\gamma = \frac{c-\gamma/2}{k_{\text{dec}}}$. We can also conclude that $x(t_{i+1})\in\Omega_{c_0}$ for all $x_i\in\Omega_{c_0}$.

Third part ($x(t_{i+1})\in\Omega_\gamma\ \forall x(t_i)\in\Omega_{\gamma/2}$*):*
This is trivially satisfied following the arguments in the first part of the proof, assuming that

$$L_\alpha\left(e^{L_{fx}\bar{\pi}}+1\right)e_{\max} \leq \gamma/2. \quad (5.41)$$

Combining the requirements of all three parts leads to the following equation that $e_{\max}$ must satisfy, such that stability is guaranteed:

$$e_{\max} \leq \frac{1}{L_\alpha \left(e^{L_{fx}\bar{\pi}} + 1\right)} \min\{c - c_0, 1/2\ \Delta\alpha_{\min}(c, \gamma/4), \gamma/4\}. \tag{5.42}$$

■

The derived result underpins that sufficiently small measurement/estimation errors can be tolerated to achieve stability in a practical sense. Thus, small measurement noise, but also state observation errors can be tolerated. The derived results lays the basis for nonlinear separation principle like results as presented in the Chapter 6.

5.7 Inherent Robustness of Sampled-data Open-loop NMPC

In the case of NMPC some inherent robustness results already exist (Magni and Sepulchre, 1997; Chen and Shaw, 1982; Mayne et al., 2000; Scokaert et al., 1997). However, these results are either only valid for instantaneous NMPC (Magni and Sepulchre, 1997; Chen and Shaw, 1982; Mayne et al., 2000), or discrete time NMPC (Scokaert et al., 1997), or they consider special NMPC implementations, such as dual-mode predictive control (Michalska and Mayne, 1993) or contractive predictive control formulations (de Oliveira Kothare and Morari, 2000; Yang and Polak, 1993).

The inherent robustness results derived in this section are also applicable to NMPC. As outlined in Section 4.4.2, many NMPC approaches that guarantee stability already satisfy the decrease condition (5.5) of Assumption 5.3. However, it is in general not possible to answer the question if a given sampled-data open-loop NMPC schemes satisfies the Lipschitz condition on the decreasing function, which in the case of NMPC is the value function. This problem stems from the fact that the applied input is based on the solution of an optimal control problem, which can be discontinuous as a function of the considered state. While this is for example of advantage in the case of the stabilization of systems that require a discontinuous input (Meadows et al., 1995; Fontes, 2000b; Fontes, 2003), and thus possibly lead to a discontinuous value function, it is of disadvantage with respect to the inherent robustness considerations in this chapter.

There are only a few NMPC schemes that guarantee that the value function is locally Lipschitz. Most of these do not consider constraints and are based on control Lyapunov function considerations, see e.g. (Jadbabaie et al., 2001). Also in the case that the system is linear and that only linear constraints are present, it is well known that the resulting value function is locally Lipschitz.

While for small systems the question if the value function is Lipschitz might be verified by numerically calculating the value function and checking approximately the local Lipschitz property, this is certainly not practicable for large systems. Thus, further research is needed investigating the question when the value function is Lipschitz, or how this, even in the presence of constraints could be ensured. Starting points might be the discrete time considerations presented in (Grimm et al., 2003b).

The derived results are still applicable to NMPC, at least locally, since many sampled-data open-loop feedback schemes, satisfy the local Lipschitz assumption of the value function locally around the origin (Chen and Allgöwer, 1998b; Chen et al., 2000; Mayne et al., 2000; Findeisen and Allgöwer, 2001).

5.8 Summary

In this chapter we derived inherent robustness properties for sampled-data open-loop feedbacks. Specifically, we showed that if the decreasing function of the sampled-data open-loop feedback is continuous, then the closed-loop possesses inherent robustness properties with respect to additive disturbances, disturbances in the input signal, and disturbances/measurement errors in the state that is used to calculate the open-loop input signal. While the derived results can in general not be used for the design of robustly stabilizing feedbacks, they underpin that sampled-data open-loop feedbacks can reject certain disturbances while guaranteeing practical stability. The main limitation of the derived results is that they require that the decreasing function is locally Lipschitz over the region of interest. While for sampled-data open-loop feedbacks stemming from the feedforward simulation of an instantaneous locally Lipschitz feedback this condition is often satisfied, in the case of sampled-data open-loop NMPC it is often not possible to provide a clear answer of whether the value function is locally Lipschitz. However, many NMPC schemes at least satisfy the continuity assumption locally around the origin. In the case of a continuous value function the derived results underpin that sampled-data open-loop NMPC can reject a series of practically relevant disturbances. Of special interest is the robustness with respect to numerical errors in the solution of the optimal control problem, and the robustness with respect to state measurement/estimation errors.

We utilize the results on the robustness with respect to state estimation errors in the next chapter to derive stability results for the sampled-data open-loop output-feedback problem.

Chapter 6

Sampled-data Open-loop Output-feedback

The results on nominal stability and on inherent robustness of sampled-data open-loop feedback are based on the assumption that the full state information is available. In practical applications it is, however, often not possible to measure all states. In practice the problem of output-feedback is often "solved" according to the certainty equivalence principle, i.e. instead of the true, but unknown, system state, a state estimate provided by a state observer is used for feedback. This often leads to good performance of the closed-loop. However, since no general separation principle for nonlinear systems exists, the stability of the closed-loop cannot be deduced from the stability of the observer and the state-feedback separately. In this chapter we derive conditions ensuring semi-regional practical stability of the closed-loop for a broad class of sampled-data open-loop feedback controllers and state observer.

The derived results are inspired by special nonlinear separation principles for instantaneous feedbacks employing high-gain observers, see e.g. (Esfandiari and Khalil, 1992; Teel and Praly, 1995; Atassi and Khalil, 1999; Maggiore and Passino, 2004; Maggiore and Passino, 2003; Shim and Teel, 2001; Shim and Teel, 2003). The state estimation error is basically considered as a disturbance acting on the nominal closed-loop and it is shown that if the sampled-data feedback possesses inherent robustness properties and if the observer error converges sufficiently fast, it is possible to achieve stability.

This chapter is structured as follows: In Section 6.1 we present the considered setup. Section 6.2 contains the main results, explicit stability conditions on the sampled-data open-loop feedback controller and the state observer such that the closed-loop is semi-regionally practically stable. Furthermore, comments with respect to the achieved results for sampled-data open-loop NMPC are given. Section 6.3 comments on some of the observers satisfying the required conditions. The derived results are exemplified in Section 6.4 considering two examples, the control of a pendulum-cart system and the control of a mixed-culture bioreactor.

The presented results are generalizations of the output-feedback results for NMPC as presented in (Imsland, Findeisen, Bullinger, Allgöwer and Foss, 2003; Findeisen et al., 2003b; Findeisen et al., 2003d; Findeisen et al., 2003c; Findeisen, Imsland, Allgöwer and Foss, 2002).

6.1 Setup

We consider the stabilization of time-invariant nonlinear systems of the form

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \tag{6.1a}$$

$$y(t) = h(x(t), u(t)) \tag{6.1b}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the input vector, and $y(t) \in \mathbb{R}^p$ are the measured outputs. As before, besides stabilization we require the input to satisfy the input constraints $u \in \mathcal{U} \subset \mathbb{R}^p$ and the states to stay in an admissible set $\mathcal{X} \subseteq \mathbb{R}^n$. We furthermore assume the following:

Assumption 6.1 $(0,0) \in \mathcal{X} \times \mathcal{U}$.

Assumption 6.2 *The vector field $f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$ is locally Lipschitz in u and x, with $f(0,0) = 0$.*

The input to the system (5.3) is given by a sampled-data open-loop feedback controller

$$u(t) = u_{SD}(t; \hat{x}(t_i), t_i). \tag{6.2}$$

Here $\hat{x}(t_i)$ is the estimated state provided by the used state observer. In the following we denote the state estimation error by $e = x - \hat{x}$.

With respect to the sampled-data open-loop feedback u_{SD} we assume slightly modified conditions, in comparison to the requirements of Theorem 5.3:

Assumption 6.3 (Conditions on the sampled-data open-loop feedback)

1. *The input generator u_{SD} is admissible with respect to a set $\mathcal{R}$, the input and state constraint sets $\mathcal{U}$, $\mathcal{X}$, and the partition π.*
2. *For all $x(t_i) \notin \mathcal{R}$, $t_i \in \pi$ the sampled-data open-loop feedback u_{SD} is defined as*

$$u_{SD}(\tau; x(t_i), t_i) = u_c, \qquad \tau \in [t_i, t_{i+1}], \tag{6.3}$$

   where $u_c \in \mathcal{U}$ is constant.

3. *There exists a locally Lipschitz continuous positive definite function $\alpha : \mathcal{R} \to \mathbb{R}^+$ and a continuous positive definite function $\beta : \mathcal{R} \to \mathbb{R}^+$, such that for all $t_i \in \pi$, $x(t_i) \in \mathcal{R}$ and $\tau \in [t_i, t_{i+1})$*

   (a)
$$\alpha(x(t_i + \tau; x(t_i), u_{SD}(\cdot; x(t_i), t_i))) - \alpha(x(t_i)) \le -\int_{t_i}^{t_i+\tau} \beta(x(s; x(t_i), u_{SD}(\cdot; x(t_i), t_i)))ds \tag{6.4}$$

   holds.

   (b) *for all compact strict subsets $\mathcal{S} \subset \mathcal{R}$ there is at least one level set $\Omega_c = \{x \in \mathcal{R} | \alpha(x) \le c\}$ s.t. $\mathcal{S} \subset \Omega_c$.*

The additional Assumption 6.3.2. is necessary, since the state estimate of the observer can be outside of $\mathcal{R}$, at least in some initial phase.

In the next section we derive semi-regional practical stability assuming that after an initial phase the observer error at the recalculation instants can be made sufficiently small. Therefore we assume the following:

Assumption 6.4 *(***Observer error convergence)**
For any desired maximum state estimation error $e_{\max} > 0$ there exist observer parameters such that

$$\|x(t_i) - \hat{x}(t_i)\| \leq e_{\max}, \qquad \forall t_i \geq t_{k_{\text{conv}}}. \tag{6.5}$$

Here $k_{\text{conv}} > 0$ is a freely chosen, but fixed number of recalculation instants after which the observer error has to satisfy (6.5).

Remark 6.1 *Depending on the observer, further conditions on the system might be necessary (e.g. observability assumptions). Also, note that the observer does not have to operate continuously since the state information is only required at the recalculation instants t_i. Thus, it is in principle possible to apply a discrete time observer for the continuous time system, or a state estimator utilizing a certain piece of the output trajectory at once, such as moving horizon state estimation (Michalska and Mayne, 1995; Zimmer, 1994; Alamir, 1999; Rao et al., 2003).*

In principle we follow the ideas used for inherent robustness with respect to measurement errors in the previous chapter, i.e. we show that if $e_{\max}$ is sufficiently small, then a decrease of the disturbed decreasing function α from recalculation time to recalculation time can be retained. However, in comparison to the previous results we must take into account that the observer requires a certain convergence time to achieve the desired maximum observer error $e_{\max}$. To avoid that the system state leaves the set Ω_c during this time it might thus be necessary to sufficiently decrease the maximum recalculation time $\bar{\pi}$.

6.2 Semi-regional Practical Stability

Under the given setup the following theorem holds

Theorem 6.1 (Semi-regional practical stability of sampled-data open-loop output-feedback)
Given level sets Ω_γ, Ω_c, and Ω_{c_0} with $\Omega_\gamma \subset \Omega_{c_0} \subset \Omega_c \subset \mathcal{R}$. Then, under the Assumptions 6.1- 6.4, there exists a maximum allowable observer error $e_{\max}$ and a maximum recalculation time $\bar{\pi}$ such that for all initial conditions $x_0 \in \Omega_{c_0}$ the state trajectories of the closed-loop

$$\dot{x}(t) = f(x(t), u_{SD}(t; \hat{x}(t_i), t_i)), \quad x(0) = x_0 \tag{6.6a}$$

$$y(t) = h(x(t), u(t)), \tag{6.6b}$$

satisfy $x(\tau) \in \Omega_c$ $\tau \geq 0$, and there exists a finite time T_γ such that $x(\tau) \in \Omega_\gamma$ $\forall \tau \geq T_\gamma$.

Remark 6.2 *We limit the consideration to level sets for the desired set of initial conditions (Ω_{c_0}), the maximum attainable set (Ω_c) and the set of desired convergence (Ω_γ) for simplification of the presentation only. In principle, one can consider arbitrary compact sets containing the origin which are subsets of each other and of $\mathcal{R}$. Assumption 6.3.3 (b) assures that it is always possible to find suitable covering level sets in this case .*

Proof: The proof is similar to the proof of Theorem 5.3. The first part ensures that the system state does not leave the maximum admissible set Ω_c during the convergence time $t_{k_{\text{conv}}}$ of the observer. This is achieved by sufficiently decreasing the maximum recalculation time $\bar{\pi}$. In the second part is shown that for a sufficiently small $e_{\max}$ the system state converges to the set $\Omega_{\gamma/2}$. The third part establishes that the state does not leave the set Ω_γ once it has entered it at a recalculation time. For the derivations we use the same notation as in the proof of Theorem 5.3.

First part ($x(t_i+\tau)\in\Omega_c\ \forall x(t_i)\in\Omega_{c_0}$):
Note that Ω_{c_0} is strictly contained in Ω_c and thus also in Ω_{c_1}, with $c_1 = c_0 + (c - c_0)/2$. Thus, there exists a time T_{c_1} such that $x(\tau) \in \Omega_{c_1}$, $\forall 0 \leq \tau \leq T_{c_1}$. The existence is guaranteed, since as long as $x(t) \in \Omega_c$, $\|x(t) - x_0\| \leq \int_0^t \|f(x(s), u(s))\|ds \leq k_{\Omega_c} t$, where k_{Ω_c} is a constant depending on the Lipschitz constants of f and the bounds on u. We take T_{c_1} as the smallest (worst case) time to reach the boundary of Ω_{c_1} from any point $x_0\in\Omega_{c_0}$ allowing $u(s)$ to take any value in $\mathcal{U}$. We pick the maximum recalculation time $\bar{\pi}$ such that $\bar{\pi} \leq T_{c_1}/k_{\text{conv}}$, it is fixed for the remainder of the proof. By Assumption 6.4 there always exist observer parameters such that after this time the observer error is smaller than any desirable $e_{\max}$.

As for the proof of Theorem 5.3, the following equality is valid as long as the states stay within Ω_c:

$$\begin{aligned}\alpha(x(\tau;x_i,u_{\hat{x}}))-\alpha(x_i) = \alpha(x(\tau;x_i,u_{\hat{x}})) - \alpha(x(\tau;\hat{x}_i,u_{\hat{x}}))\\ + \alpha(x(\tau;\hat{x}_i,u_{\hat{x}})) - \alpha(\hat{x}_i) + \alpha(\hat{x}_i) - \alpha(x_i).\end{aligned} \tag{6.7}$$

One way to ensure that $\hat{x}_i\in\Omega_c$ if $x_i\in\Omega_{c_1}$ is to require that $L_\alpha e_{\max} \leq c - c_1$. Thus, we obtain:

$$\alpha(x(t_i + \tau;x_i,u_{\hat{x}})) - \alpha(x_i) \leq L_\alpha e^{L_{fx}\tau}\|e_i\| - \int_{t_i}^{t_i+\tau}\beta(x(s;\hat{x}_i,u_{\hat{x}}))ds + \alpha_{\mathcal{K},\Omega_c}(\|e_i\|) \tag{6.8}$$

As the contribution of the integral is negative, requiring

$$L_\alpha(e^{L_{fx}\bar{\pi}} + 1)e_{\max} \leq c - c_1, \tag{6.9}$$

yields $x(t_i + \tau)\in\Omega_c\ \forall\tau\in(t_{i+1} - t_i)$.

The second and third part of the proof are the same as for Theorem 5.3. From the combination of all three parts we finally obtain that if

$$\bar{\pi} \leq T_{c_1}/k_{\text{conv}} \tag{6.10}$$

and if we choose the maximum observer error $e_{\max}$ such that

$$e_{\max} \leq \frac{1}{L_\alpha(e^{L_{fx}\bar{\pi}} + 1)}\min\{c - c_1, \frac{1}{2}\Delta\alpha_{\min}(c,\gamma/4),\gamma/4\}, \tag{6.11}$$

then for all $x_0\in\Omega_{c_1}$: $x(\tau)\in\Omega_c\ \tau \geq 0$, and there exists a finite time T_γ such that $x(\tau)\in\Omega_\gamma\ \forall\tau \geq T_\gamma$.■

The most critical conditions for the application of the derived semi-regional practical stability result is the requirement that the observer satisfies Assumption 6.4. Even so this assumption is rather strong, a series of observer designs exist achieving the desired properties, as described in Section 6.3.

In the spirit of the results presented in (Teel and Praly, 1995; Atassi and Khalil, 1999) it can be argued that the derived results are a special separation principle for sampled-data open-loop output-feedback. However, one should note that in comparison to the linear separation principle, the observer and controller design are not completely independent. Specifically, the speed of convergence of the observer must be sufficiently high. Also, the recalculation time cannot be freely chosen. It rather must be sufficiently small to avoid the system state leaving the admissible set in the initial phase, in which the observer error has not converged.

As for the inherent robustness of sampled-data open-loop feedback, the derived bounds can in general not be used for the design of a suitable observer and feedback. The result rather underpins that if the observer error can be decreased sufficiently fast, then the closed-loop system is semi-regional practical stable.

Remark 6.3 *(***Recovery of performance)** *If the input generator provides input trajectories depending continuously on the state, it is possible to show that the performance of the state-feedback is recovered as* $\bar{\pi} \to 0$ *and* $e_{\max} \to 0$*. For a derivation of this result for nonlinear predictive control see (Imsland, Findeisen, Bullinger, Allgöwer and Foss, 2003).*

Remark 6.4 (Remarks on output-feedback sampled-data open-loop NMPC) *As for the inherent robustness results, the main limitation of the derived results with respect to sampled-data open-loop NMPC is the uniform local Lipschitz requirement on the decreasing function. As mentioned in Section 5.7, this property is often difficult to verify, especially since the value function is typically not known explicitly. In the case of NMPC the derived result have, in comparison to other output-feedback results (Scokaert et al., 1997; Magni, De Nicolao and Scattolini, 2001a; de Oliveira Kothare and Morari, 2000), the advantage that the obtained stability is semi-regional rather then local.*

6.3 Suitable Observer Designs

Satisfying Assumption 6.4 is in general difficult. However, by now there exist a series of observer designs satisfying Assumption 6.4. Examples are high-gain observers (Tornambè, 1992), optimization based moving horizon observers with contraction constraint (Michalska and Mayne, 1995), observers possessing a linear error dynamics where the poles can be chosen arbitrarily (e.g. based on normal form considerations and output injection (Bestle and Zeitz, 1983; Krener and Isidori, 1983)), and observers achieving finite convergence time such as sliding mode observers (Drakunov and Utkin, 1995) or the approach presented in (Engel and Kreisselmeier, 2002; Menold et al., 2003). We shortly provide some more details on high-gain observers and moving horizon observers.

6.3.1 High Gain Observers

One possible observer class satisfying Assumption 6.4 are high-gain observers. Basically, high-gain observers obtain a state estimate based on approximated derivatives of the output signals. They are in general based on the assumption that the system is uniformly completely observable. Uniform complete observability is defined in terms of the observability map $\mathcal{H}$, which is given by successive differentiation of the output y (assuming f and h, as well as the input are sufficiently often differentiable):

$$\begin{aligned} H(x,U) &= \left[y_1,\dots y_1^{(r_1)}, y_2,\dots,y_p^{(r_p)}\right]^T \\ &= \left[h_1(x,u),\dots,\psi_{1,r_1}(x,u,\dot{u},\dots,u^{(r_1)}), h_2(x,u),\dots,\psi_{p,r_p}(x,u,\dot{u},\dots,u^{(r_p)})\right]^T. \end{aligned}$$

Here $\sum_{i=1}^{p}(r_i+1)=n$, and $U=[u_1,\dot{u}_1,\dots,u_1^{(m_1)},u_2,\dot{u}_2,\dots,u_m,\dot{u}_m,\dots,u_m^{(m_m)}]^T\in\mathbb{R}^{m_U}$ where the m_i denote the number of really necessary derivatives of the input i with $m_U=\sum_{i=1}^{m}(m_i+1)$. The $\psi_{i,j}$'s are defined via the successive differentiation of y

$$\psi_{i,0}(x,u) = h_i(x,u), \quad i=1,\dots,p \tag{6.12a}$$

$$\psi_{i,j}(x,u,\dots,u^{(j)}) = \frac{\partial\psi_{i,j-1}}{\partial x}\cdot f(x,u) + \sum_{k=1}^{j}\frac{\partial\psi_{i,j-1}}{\partial u^{(k-1)}}\cdot u^{(k)}, \quad i=1,\dots,p, \;\; j=1,\dots,r_p. \tag{6.12b}$$

Note that in general, not all derivatives of the u_i up to order $\max\{r_1,\dots,r_p\}$ appear in $\psi_{i,j}$.

Uniform complete observability basically ensures the existence of an invertible observability map (Tornambè, 1992; Teel and Praly, 1995; Shim and Teel, 2003):

Definition 6.1 (Uniform Complete Observability)
The system (2.1) *is uniformly completely observable if there exists a set of indices* $\{r_1,\dots,r_p\}$ *such that the mapping defined by* $Y=\mathcal{H}(x,U)$ *is smooth with respect to* x *and its inverse from* Y *to* x *is smooth and onto for any* U*.*

The inverse of $\mathcal{H}$ with respect to x is denoted by $\mathcal{H}^{-1}(Y,U)$, that is $x=\mathcal{H}^{-1}(Y,U)$.

Remark 6.5 *In general the set of indices* $\{r_1,\dots,r_p\}$ *is not unique, different* $\mathcal{H}$ *might exist. One can use this degree of freedom to find a* $\mathcal{H}$ *such that only a minimum number of derivatives of* u*, possibly none, are necessary. This is desirable, since all inputs* u *and all the derivatives of* u *that appear in* U *must be known.*

For simplicity of presentation we assume in the following that the observability map does not depend on the input and its derivatives:

Assumption 6.5 $\mathcal{H}(x,U)=\mathcal{H}(x)$.

More general results specifically considering sampled-data open-loop NMPC allowing the observability map $\mathcal{H}$ to depend on the input and its derivatives can be found in (Findeisen et al., 2003b).

Applying the coordinate transformation $z = \mathcal{H}(x)$ leads to the system in observability normal form in z coordinates

$$\dot{z} = Az + B\phi(z,u), \qquad y = Cz. \tag{6.13}$$

The matrices A, B and C have the following structure

$$A = \text{blockdiag}\left[A_1, \ldots A_p\right], \ A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{r_i \times r_i} \tag{6.14a}$$

$$B = \text{blockdiag}\left[B_1, \ldots, B_p\right], B_i = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T_{r_i \times 1} \tag{6.14b}$$

$$C = \text{blockdiag}\left[C_1, \ldots, C_p\right], C_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}_{1 \times r_i}, \tag{6.14c}$$

and $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is the "system nonlinearity" in observability normal form. In these coordinates the high-gain observer

$$\dot{\hat{z}} = A\hat{z} + H_\epsilon(y - C\hat{z}) + B\hat{\phi}(\hat{z},u), \quad z(0) = z_0 \tag{6.15}$$

allows recovery of the states (Tornambè, 1992; Atassi and Khalil, 1999) z from information of $y(t)$ only, provided that $\hat{\phi}$ in (6.15) is globally bounded. The function $\hat{\phi} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is the approximation of ϕ used in the observer. The observer gain matrix H_ϵ is given by $H_\epsilon = \text{blockdiag}\left[H_{\epsilon,1}, \ldots, H_{\epsilon,p}\right]$, with $H_{\epsilon,i}^T = [\alpha_1^{(i)}/\epsilon, \ \ \alpha_2^{(i)}/\epsilon^2, \ldots, \alpha_{r_i}^{(i)}/\epsilon^{r_i}]$, where ϵ is the so-called high-gain parameter since $1/\epsilon$ goes to infinity for $\epsilon \to 0$. The $\alpha_j^{(i)}$s are design parameters and must be chosen such that the polynomials

$$s^{r_i} + \alpha_1^{(i)} s^{r_i - 1} + \cdots + \alpha_{r_i - 1}^{(i)} s + \alpha_{r_i}^{(i)} = 0, \ i = 1, \ldots, p \tag{6.16}$$

are Hurwitz. Note that estimates obtained in z coordinates can be transformed back to the x coordinates by $\hat{x} = \mathcal{H}^{-1}(\hat{z})$.

As shown in (Atassi and Khalil, 1999), under the assumption that the initial observer error is out of a compact set and that the system state stays in a bounded region, for any desired $e_{\max}$ and any convergence time $t_{k_{\text{conv}}}$ there exists a maximum $\epsilon^\star$ such that: for any $\epsilon \leq \epsilon^\star$ the observer error stays bounded and satisfies: $\|x(\tau) - \hat{x}(\tau)\| \leq e_{\max} \ \forall \tau \geq t_{k_{\text{conv}}}$. Thus, the high-gain observer satisfies Assumption 2. Further details, specifically considering sampled-data open-loop NMPC as input generator allowing $\mathcal{H}$ to depend on the input and its derivatives can be found in (Findeisen et al., 2003b; Findeisen, Imsland, Allgöwer and Foss, 2002).

Remark 6.6 (Possible expansions and generalizations) *If the input and its derivatives appear in the observability map, it is necessary to require that the input provided by the input generator must be sufficiently smooth. Furthermore, since in general the input jumps at the recalculation instants, it might be necessary to reset the observer to avoid jumps and divergence of the state estimates, compare (Findeisen et al., 2003b; Findeisen, Imsland, Allgöwer and Foss, 2002). Moreover it is possible to formulate the high gain observer purely in the original coordinates, thus avoiding explicit knowledge of the inverse of the observability map (Maggiore and Passino, 2000; Findeisen et al., 2003a). In (Imsland, Findeisen, Allgöwer and Foss, 2003a; Imsland, Findeisen, Allgöwer and Foss, 2003b), conditions on the system and the observer are given for the state to actually converge to the origin.*

6.3.2 Moving Horizon Observers

Moving horizon estimators (MHE) are optimization-based observers. The state estimate is obtained by a dynamic optimization problem minimizing the deviation between the measured output and the simulated output starting from an estimated initial state at time $t - T$, where T is the window length used for the state estimation. Various moving horizon state estimation approaches exist (Michalska and Mayne, 1995; Zimmer, 1994; Alamir, 1999; Rao et al., 2003). We focus here on the MHE scheme with contraction constraint as introduced in (Michalska and Mayne, 1995) since it satisfies the assumptions needed. This approach basically proposes to solve at all recalculation instants a dynamic optimization problem, considering the output measurements spanning over a certain estimation window in the past. Assuming certain reconstructability assumptions to hold and that no disturbances are present, one can, in principle, estimate the system state exactly by solving one single dynamic optimization problem. However, since this involves the solution of a global optimization problem in real-time, it is proposed in (Michalska and Mayne, 1995) to only improve the estimate at every recalculation time requiring the integrated error between the measured output and the simulated output is decreasing from recalculation instant to recalculation instant. Since the contraction rate directly corresponds to the convergence of the state estimation error and since it can in principle be freely chosen this MHE scheme satisfies all required assumptions. Thus, it can be used together with a state-feedback NMPC controller to achieve semi-regional practical stability.

6.4 Examples

In this section, we exemplify the derived result considering the control of a pendulum-cart system and of a mixed-culture bioreactor using high-gain observers for state recovery.

6.4.1 Example I: Control of a Bioreactor

We consider the control of a continuous mixed culture bioreactor as presented in (Hoo and Kantor, 1986). Schematics of the considered process are shown in Figure 6.1. The bioreactor contains a

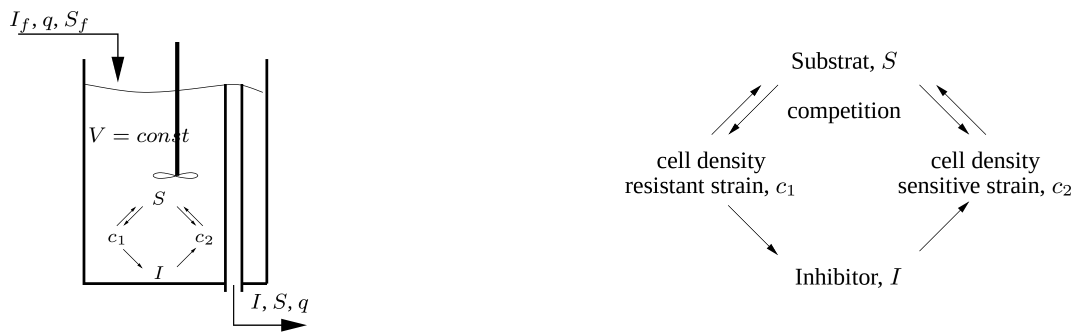


Figure 6.1: Schematic diagram of the continuous mixed culture bioreactor and the strain/inhibitor interactions.

culture of two cell strains, in the following called species 1 and 2 having different sensitivity to an external growth-inhibiting agent. The interactions of the two cell populations are illustrated in the right part of Figure 6.1. The cell density of the inhibitor resistant strain is denoted by c_1, the cell density of the inhibitor sensitive strain is denoted by c_2, and the substrate and inhibitor concentrations in the reactor are denoted by S and I. Based on the full model described in (Hoo and Kantor, 1986) a reduced third order model of the following form can be obtained

$$\frac{dc_1}{dt} = \mu_1(S)c_1 - c_1 u_1, \tag{6.17}$$

$$\frac{dc_2}{dt} = \mu_2(S, I)c_2 - c_2 u_1, \tag{6.18}$$

$$\frac{dI}{dt} = -pc_1 I + u_2 - I u_1. \tag{6.19}$$

The inputs are the dilution rate u_1 and the inhibitor addition rate u_2. The deactivation constant of the inhibitor for species 2 is denoted by p. The specific growth rates $\mu_1(S)$ and $\mu_2(S, I)$ are given by

$$\mu_1(S) = \frac{\mu_{1,M} S}{K + S}, \quad \mu_2(S, I) = \frac{\mu_{2,M} S}{K + S} \frac{K_I}{K_I + I}. \tag{6.20}$$

where K, K_I, $\mu_{1,M}$ and $\mu_{2,M}$ are constant parameters, see Table 6.1. The substrate concentration is

Table 6.1: Parameters of the bioreactor model.

Parameter	Value	Parameter	Value
$\mu_{1,M}$	0.4hr^{-1}	$\mu_{2,M}$	0.5hr^{-1}
K	0.05g/l	K_I	0.02g/l
Y_1	0.2	Y_2	0.15
S_f	2.0g/l	p	0.5hr^{-1}/g

given by

$$S = S_f - \frac{c_1}{Y_1} - \frac{c_2}{Y_2}. \tag{6.21}$$

Here Y_1, Y_2 are the yields of the species and S_f is the substrate inlet concentration. The control objective is to stabilize the steady-state $c_{1s} = 0.016$g/l, $c_{2s} = 0.06$g/l, $I_s = 0.005$g/l. The outputs available for feedback are $y_1 = \ln \frac{c_1}{c_2}$, which can be thought of as a turbidity measurement and the cell density of species one, $y_2 = c_1$. Performing the following coordinate transformation

$$z_1 = \ln \frac{c_1}{c_2}, \quad z_2 = \mu_1(S) - \mu_2(S, I), \quad z_3 = c_1, \tag{6.22}$$

we obtain the system in observability normal form:

$$\dot{z}_1 = z_2, \tag{6.23a}$$

$$\dot{z}_2 = \phi(z, u_1, u_2), \tag{6.23b}$$

$$\dot{z}_3 = \psi(z, u_1, u_2), \tag{6.23c}$$

$$y_1 = z_1 \tag{6.23d}$$

$$y_2 = z_3 \tag{6.23e}$$

The states z_1 and z_2 are estimated from the output measurement y_1 via a high-gain observer as described in Section 6.3.1. The parameters α_1 and α_2 in the high-gain observer (6.15) are chosen to $\alpha_1 = \sqrt{2}$, $\alpha_2 = 1$. The state z_3 is not estimated, since it is directly available by measurement y_2.

As state-feedback NMPC scheme, quasi-infinite horizon NMPC (Chen and Allgöwer, 1998a) is used. The cost F weighs the quadratic deviation of the states and inputs in the new coordinates from their desired steady-state values. For simplicity, unit weights on all states and inputs are considered. A quadratic upper bound E on the infinite horizon cost and a terminal region $\mathcal{E}$ satisfying the assumptions of (Chen and Allgöwer, 1998a) are calculated using LMI/PLDI-techniques (Boyd et al., 1994). The piecewise linear differential inclusion (PLDI) representing the dynamics in a neighborhood of the origin is found using the methods described in (Slupphaug et al., 2000). The recalculation instants are equidistant, i.e. $\delta^r = 2$hrs, while the prediction horizon T_p is chosen to be $T_p = 15$hrs. The optimal input at every recalculation instant is obtained via a direct solution approach implemented in Matlab, where the input is discretized as piecewise constant, with 10 control intervals per recalculation interval.

To illustrate the stability and performance of the closed-loop, we consider different observer gains ϵ while keeping (the sufficiently small) recalculation time δ^r constant. In all simulations, the observer is initialized with the correct values for z_1, since this can be directly obtained from the measurements, whereas z_2 is initialized with the steady-state value. Figure 6.2 exemplary shows closed-loop system trajectories projected onto the c_1/c_2 phase plane for different observer gains $k = \frac{1}{\epsilon}$ in comparison to the state-feedback NMPC controller starting from the same initial condition. Figure 6.3 shows

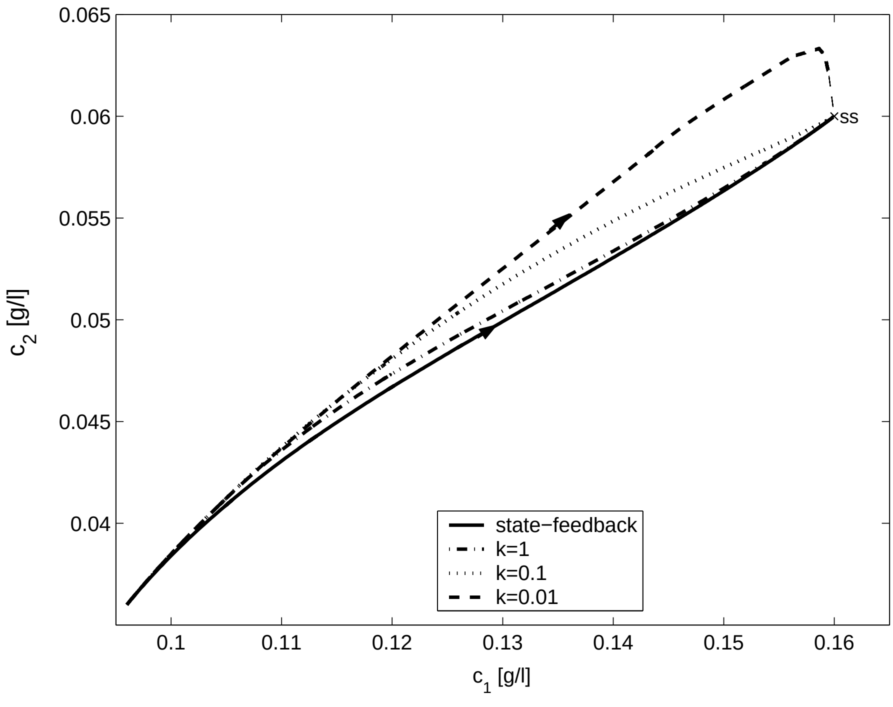


Figure 6.2: Phase plot of c_1 and c_2. SS denotes the desired steady-state.

the corresponding plots of the inhibitor concentration I and the inhibitor addition rate (input u_2) for different values of ϵ. Additionally, the real cost occurring, i.e. the integrated quadratic error between the steady-state values for the states and inputs in transformed coordinates, is plotted. The cost of

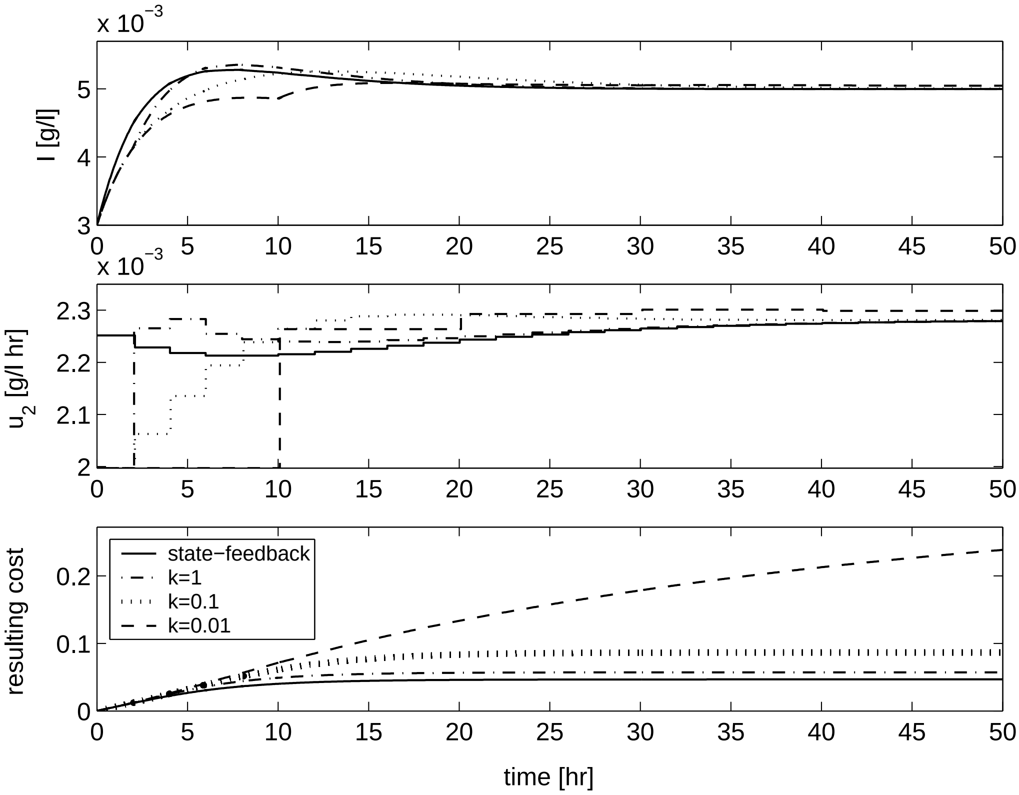


Figure 6.3: Trajectories of I, u_2 and summed up cost for different values of the high-gain parameter ϵ starting from the same initial conditions for the system and observer.

the output-feedback controller approaches the cost of the state-feedback controller for decreasing ϵ, which shows the recovery of performance. We use relatively low gains for the observer. This example verifies the stability of the closed-loop and the recovery of performance for increasing values of the observer gain.

6.4.2 Example II: Control of a Pendulum-cart System

As second example we consider the control of an (unstable) inverted pendulum on a cart, see Figure 6.4. The angle of the pendulum with the vertical axis is denoted by z_1. The input to the system is given by the force u acting on the cart's translation. It is constrained to $-10\text{N} \leq u(t) \leq 10\text{N}$. The control objective is to stabilize the angle $z_1 = 0$ (upright position) while the cart's position is not constrained (and thus not modeled nor controlled). It is assumed that only the angle z_1 but not the angular velocity can be measured. The model of the system is given by the following equations (Imsland,

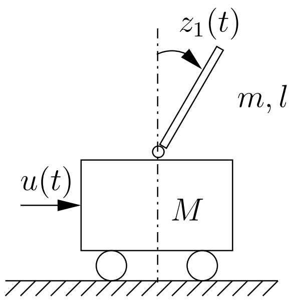


Figure 6.4: Pendulum on a cart.

Findeisen, Bullinger, Allgöwer and Foss, 2003):

$$\dot{z}_1 = z_2, \tag{6.24a}$$

$$\dot{z}_2 = \frac{ml\cos(z_1)\sin(z_1)z_2^2 - g(m+M)\sin(z_1) + \cos(z_1)u}{ml\cos^2(z_1) - \frac{4}{3}(m+M)l}, \tag{6.24b}$$

$$y = z_1, \tag{6.24c}$$

where z_2 is the angular velocity of the pendulum. The parameters $M = 1$kg, $m = 0.2$kg, $l = 0.6$m and $g = 10\frac{\text{m}}{\text{s}^2}$ are constant.

The stage cost is quadratic and the weights on the states and input are chosen as unit weights for simplicity, i.e. $F(z,u) = z^\top \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] z + u^2$. Quasi-infinite horizon NMPC (Chen and Allgöwer, 1998b; Findeisen and Allgöwer, 2001) is used as state-feedback controller. The terminal penalty cost E and the terminal region $\mathcal{E}$ are obtained using LMI/PLDI-techniques. The resulting terminal penalty cost E is given by: $E(z) = z^\top \left[\begin{smallmatrix} 311.31 & 66.20 \\ 66.20 & 34.99 \end{smallmatrix}\right] z$, and the terminal region $\mathcal{E}$ is given by $\mathcal{E} = \{z \in \mathbb{R}^2 | E(z) \leq 20\}$.

The control horizon T_p is chosen to 0.5s. The recalculation time is fixed to $\delta^r = 0.05$s and the input signal for the direct solution of the optimal control problem is parameterized as piecewise constant with $\delta^s = 0.025$s. All simulations and calculations are performed in Matlab. Figure 6.5 shows the region of attraction and the contour lines of the value function of the sampled-data open-loop state-feedback controller. The plot is obtained solving the open-loop state-feedback NMPC problem for different initial conditions of z_1 and z_2.

In the output-feedback case, whenever the state estimate leaves the region of attraction of the state-feedback NMPC scheme the input is fixed to the steady-state value 0. The states z_1 and z_2 are estimated from y using a high-gain observer. Note that the system is already given in observability normal form. The observer parameters α_1 and α_2 are chosen to $\alpha_1 = 2$ and $\alpha_2 = 1$. For all simulations, the observer is started with zero initial conditions, i.e. $\hat{z}_1 = \hat{z}_2 = 0$.

Figure 6.6 shows the phase plot of the system states and the observer states of the closed-loop system for different values of the observer parameter ϵ. As expected, for decreasing values of ϵ the trajectories of the state-feedback control scheme are recovered. Comparing both plots, one sees that for $\epsilon = 0.1$, when the observer state and the real state are at the boundary of the region of attraction of the

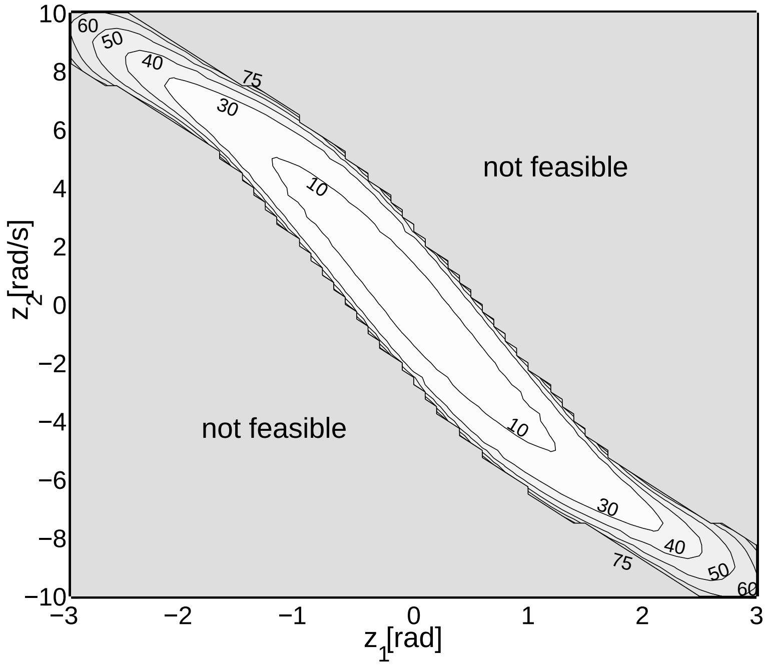
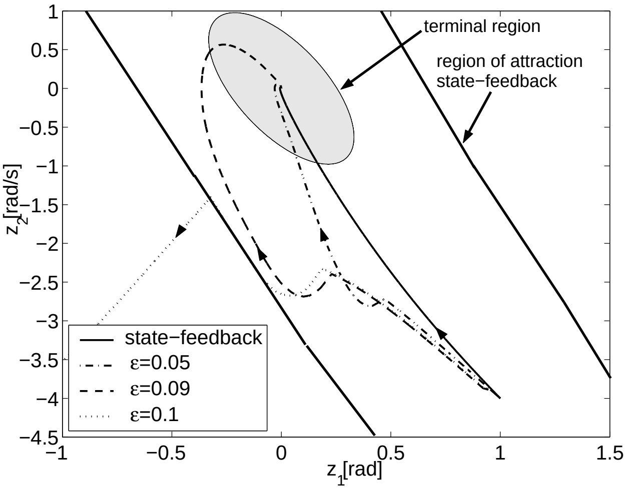


Figure 6.5: Level sets of the value function of the sampled-data open-loop state-feedback controller of the pendulum on a cart.



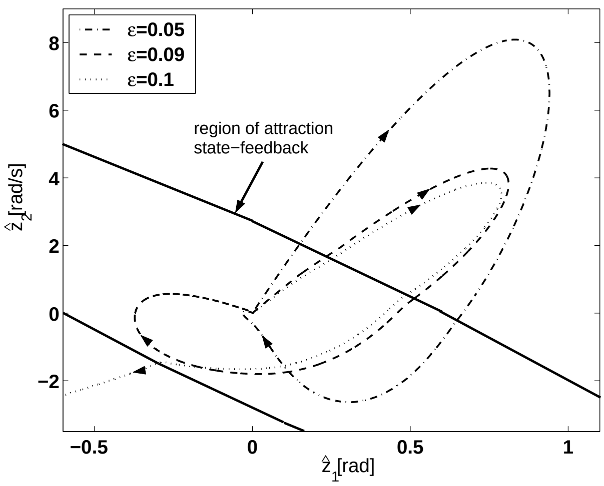


Figure 6.6: Phase plot of the nominal system states (left) and the observer states (right).

state-feedback controller, a small estimation error does lead to infeasibility of the open-loop problem and thus to divergence. For smaller values of ϵ, the correct state is recovered faster and infeasibility/divergence is avoided. However, for smaller values of ϵ a bigger (but time-wise shorter) peaking of the observer error at the beginning occurs, see Figure 6.6, right plot. This is also evident in the time plot of the states and inputs as shown in Figure 6.7. Notice also in the state-feedback case, for the initial conditions shown, the input constraints are not hit, while for all output-feedback cases the

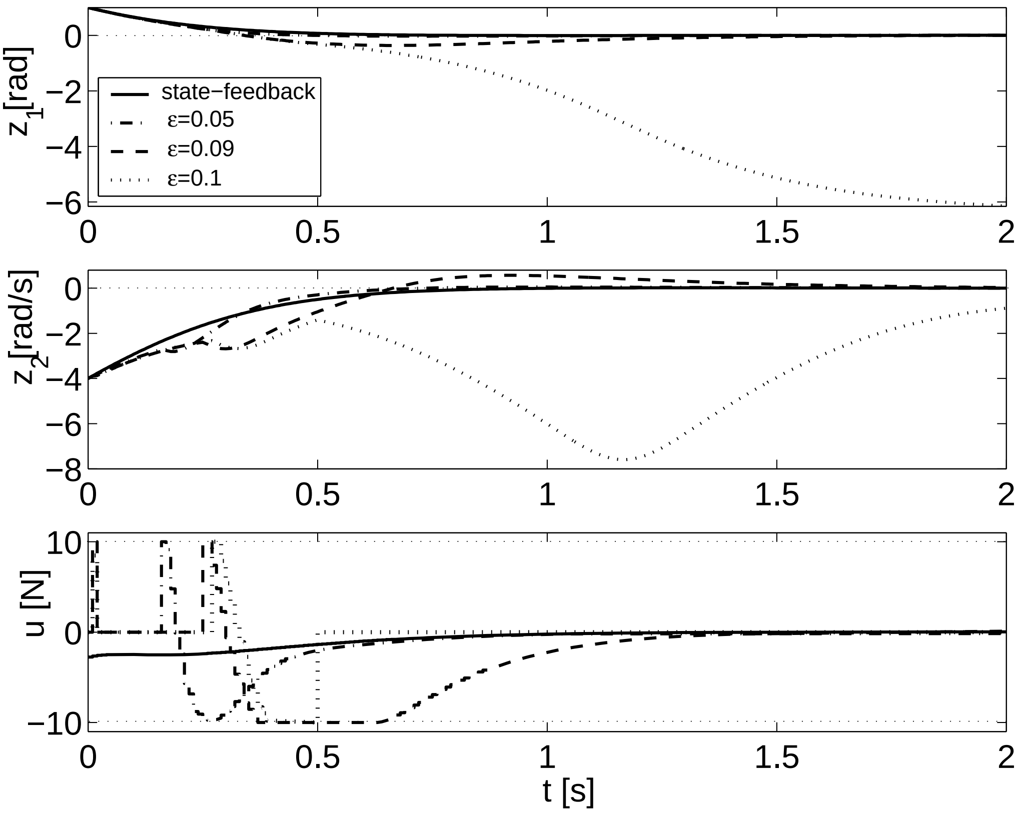


Figure 6.7: Trajectories of z_1, z_2 and the input u.

NMPC controller hits the input constraints.

Figure 6.8 shows a part of the region of attraction for different values of $\epsilon = 0.03, 0.07$ and 0.09 in comparison to the state-feedback case. Note that the region of attraction for smaller values of ϵ always contain the regions of attraction of the ones for bigger values of ϵ. The plot underpins that the region of attraction of the output-feedback controller converges to the state-feedback one for decreasing ϵ. This is in correspondence to the result of Theorem 6.1.

6.5 Summary

Deriving stabilizing output-feedback control schemes is of practical, as well as of theoretical relevance. In this chapter we outlined, based on results for NMPC (Imsland, Findeisen, Bullinger, Allgöwer and Foss, 2003; Findeisen et al., 2003b; Findeisen et al., 2003d; Findeisen et al., 2003c; Findeisen, Imsland, Allgöwer and Foss, 2002), a semi-regional practical stability result for sampled-data open-loop feedback. The result is not limited to a specific observer class, we rather state conditions on the controller and observer ensuring semi-regional practical stability. As shown, a series of observer designs satisfy the required conditions. Examples are high-gain observers, moving horizon observers with contraction constraint, observers possessing a linear error dynamics where the poles can be arbi-

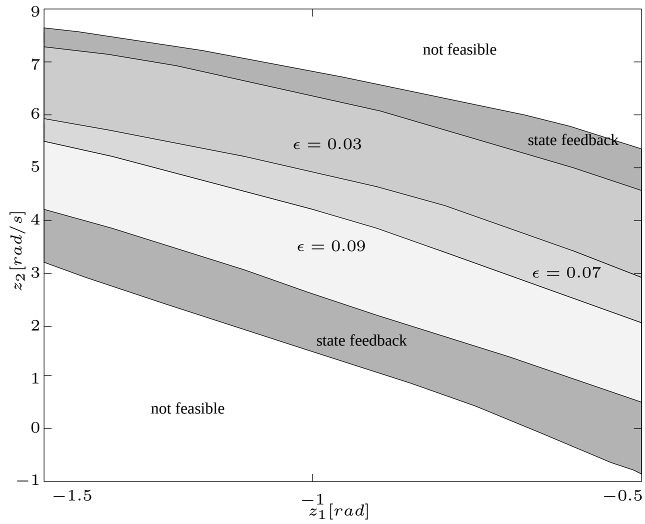


Figure 6.8: Recovery of the region of attraction.

trarily chosen, or observers achieving convergence in finite time. In the spirit of the output-feedback results for the instantaneous case as presented in (Teel and Praly, 1995; Atassi and Khalil, 1999) it could be argued that the derived results represent a separation principle for sampled-data open-loop output-feedback.

In the case of NMPC the derived result have, in comparison to other output-feedback results (Scokaert et al., 1997; Magni, De Nicolao and Scattolini, 2001a; de Oliveira Kothare and Morari, 2000), the advantage that the obtained stability is semi-regional rather then local. In comparison to (Michalska and Mayne, 1995) which present (semi-regional) closed-loop stability results for NMPC in combination with MHE, it is not necessary to find the global solution of a nonlinear dynamic optimization problem.

The price to pay is the local Lipschitz requirement on the value function. While for sampled-data open-loop feedbacks based on instantaneous locally Lipschitz continuous feedbacks, this condition is normally satisfied, in the case of sampled-data open-loop NMPC it can often not be guaranteed a priory. Thus, future research should focus on either relaxing this condition, or deriving conditions under which an NMPC scheme does satisfy this assumption, see for example (Grimm et al., 2004b).

The achieved results provide a theoretical basis for the in practice often applied certainty equivalence principle.

Chapter 7

Conclusions and Outlook

Linear model predictive control is by now one of the most widely employed advanced control techniques in industry, especially for multi-variable process control applications subject to constraints on the process variables. Steadily increasing economic and ecological demands require to operate processes over a wide range of operating conditions and close to the boundary of safe operation. For such processes, however, linear MPC often performs poorly, since a linear process model cannot capture the nonlinear dynamics sufficiently well. For this reason there is a strong interest in the development of practically applicable, reliable NMPC schemes. Even so there has been a significant progress in the area of NMPC over the recent decades, there are still many problems that must be overcome before a theoretically well founded, safe and reliable application of NMPC is possible in practice. Examples are the efficient and reliable online implementation, the analysis of the inherent robustness properties, the development of robust NMPC approaches, the compensation of delays, and the design of output-feedback NMPC approaches.

The results derived in this thesis provide answers to some of the open questions, and they provide methods that overcome some of the present shortcomings. The results are specifically focused on sampled-data open-loop NMPC implementations. Sampled-data open-loop NMPC refers to NMPC schemes, in which the optimal control problem is solved only at discrete recalculation instants, while the resulting optimal input is applied open-loop in between.

The problems considered in this thesis can be divided into two groups. Firstly, theoretical questions such as inherent robustness, delay compensation, nominal stability and output-feedback are considered. These questions must be addressed in order to lay a solid foundation for the practical application of sampled-data open-loop feedback. Secondly, issues related to the efficient and reliable implementation and solution of the open-loop optimal control problem are considered. This is one of the core elements for a practical implementation of NMPC.

With respect to the efficient solution and implementation of the occurring open-loop optimal control problem, we provide a proof of concept that even nowadays NMPC is practically applicable from a computational point of view. For this purpose we consider the control of a high-purity distillation column in simulations and experiments. The derived results underpin the two key elements related to real-time feasibility of NMPC: the application of NMPC schemes that facilitate an efficient solution and the use of specially tailored dynamic optimization approaches. Such a proof of concept of real-time implementation is important for industrial practitioners. However, the derivations and performed experiments also point to a series of open theoretical questions in the area of sampled-data open-loop NMPC: Is it possible to guarantee stability, even though numerical approximations and computational delays are present? Furthermore, in which sense do model-plant mismatch, external disturbances, and state estimation errors influence the stability and performance of the closed-loop?

To provide answers to these questions, the problem of stabilization via sampled-data open-loop feedback is considered from a general, more abstract point of view. In a first step, a new concept of stabilization based on admissible input generators is introduced. Briefly, admissible input generators provide open-loop input trajectories at the recalculation instants, based on the current state information. These input trajectories are then applied open-loop to the system until the next recalculation instant. The advantage of the outlined perception is that it allows a unified view on the problem of stabilization via sample-data open-loop feedback. In a second step, stability results for sampled-data open-loop input generators are derived. The results allow the consideration of set stabilization, input generators that might be discontinuous in the state, and state and input constraints. The derived results are exemplified considering a new sampled-data open-loop feedback strategy based on the feedforward simulation of instantaneous feedbacks. This strategy allows to adapt instantaneous, locally Lipschitz continuous state-feedbacks to the sampled-data open-loop feedback case without loss of stability.

With respect to the practical important problem of measurement, computational, as well as communication delays, we outline how these can be taken into account in sampled-data open-loop feedback control. The proposed approaches are based on a suitable shift of the input signal and on a feedforward prediction of the available state information using the available model.

Based on the derived nominal stability results the question of the inherent robustness of sampled-data open-loop feedbacks with respect to external disturbances and model-plant mismatch is considered. It is shown that under certain continuity assumptions, sampled-data open-loop feedbacks possesses inherent robustness properties. Of practical importance are the robustness to small input uncertainties such as numerical optimization errors, the robustness to small input delays, the robustness to measurement and state estimation errors, and the robustness to neglected fast actuator and sensor dynamics.

The results on inherent robustness pave the way to derive stability results for the sampled-data open-loop output-feedback problem. Specifically, for a broad class of sampled-data open-loop state-feedback controllers, including NMPC, novel conditions on the state observer are derived guaranteeing that the closed-loop is semi-regionally practically stable. Even so the required conditions on the state observer are rather stringent, a series of observer designs do satisfy them. Examples are high-gain observers, moving horizon observers subject to contraction constraints, observers possessing a linear error dynamics where the poles can be arbitrarily chosen, or observers achieving convergence in finite time such as sliding mode observers.

Overall, the results in this thesis provide answers and solutions to a series of practically and theoretically important open questions and problems for sampled-data open-loop feedback, especially

NMPC. The derived results, especially on nominal stability, inherent robustness and output-feedback provide a solid theoretical basis for the application of sampled-data open-loop NMPC under practical conditions. Notably, the majority of the derived results are not limited to sampled-data open-loop NMPC. They rather hold for a wide class of sampled-data open-loop feedbacks.

7.1 Outlook

Several results addressed in this thesis offer the opportunity for further research.

Firstly, the derived methods and results should be validated by means of practically relevant examples. Especially, the inevitable conservatism incorporated in the developed robust results and in the proposed approach towards output-feedback NMPC has to be examined concerning realistic applications. The technical machinery to facilitate such examinations can be based on the results of Chapter 5 and 6.

Another promising area for further research is the combination of a sampled-data open-loop input generator and an instantaneous feedback controller tracking the corresponding open-loop trajectory to counteract disturbances instantaneously. Such a “hybrid” control approach should in principle increase the robustness and performance of the closed-loop significantly. Preliminary results in this direction are outlined in (Lepore et al., 2004; Fontes and Magni, 2003).

With respect to the output-feedback problem for NMPC, it is of practical, as well as theoretical interest to overcome the required local Lipschitz continuity of the value function. It is in general not possible to verify the continuity requirement on the value function a priori for stabilizing MPC schemes, especially if state constraints are present. Thus, future research might focus on either relaxing this condition, or to derive conditions under which an NMPC scheme does satisfy this assumption.

One possible solution to overcome the local Lipschitz conditions is the direct consideration of an estimate of the observer error in the predictive controller, i.e. the design of an NMPC controller that explicitly accounts for the disturbance due to the state estimate. This can be addressed by set-based observers and min-max NMPC formulations (see e.g. (Findeisen and Allgöwer, 2004b; Findeisen and Allgöwer, 2004c)). However, there are many open questions concerning the efficient solution of the resulting min-max problem or the specific design of the set-based observer. The advantage of such an approach is its potential capability to stabilize systems requiring a discontinuous feedback, as for example nonholonomic mechanical systems.

Appendix A

Proof of Lemma 4.1

The proof is similar to the contradiction argument given in the proof of Lemma 4 in (Michalska and Vinter, 1994) for the non set-based case, with slight modifications as used in Theorem 14.1 of (Yoshizawa, 1966).

Proof: :
Assume the contrary. Since $\|x(\cdot)\|_{L^\infty(0,\infty)} < \infty$ and $\|\dot{x}(\cdot)\|_{L^\infty(0,\infty)} < \infty$, we know that there exist positive constants k_1, k_2 such that

$$\|x(\cdot)\|_{L^\infty(0,\infty)} \leq k_1, \quad \|\dot{x}(\cdot)\|_{L^\infty(0,\infty)} \leq k_2.$$

Then we know that there exists a positive constant c with $k_1 > c > 0$ and a sequence of times $\{t_i\}_{i\in\mathbb{N}}$, such that for $i \to \infty$ $t_i \to \infty$, and that $\|x(t_i)\|_{\mathcal{A}} \geq c$. We can extract from this sequence a subsequence $\{\tilde{t}_i\}_{i\in\mathbb{N}}$, such that for all $i \in \mathbb{N}$:

$$|\tilde{t}_{i+1} - \tilde{t}_i| > \frac{c}{2k_2}.$$

Note that this implies that the intervals $[\tilde{t}_i, \tilde{t}_i + \frac{c}{2k_2}]$ are disjoint. Furthermore, from $\|x\|_{\mathcal{A}} - \|y\|_{\mathcal{A}} \leq \|x - y\|_{\mathcal{A}}$ it follows that for $t > \tilde{t}_i$

$$\|x(\tilde{t}_i)\|_{\mathcal{A}} - \|x(t)\|_{\mathcal{A}} \leq \|x(\tilde{t}_i) - x(t)\| = \left\|\int_{\tilde{t}_i}^{t} \dot{x}(\tau)d\tau\right\| \leq \int_{\tilde{t}_i}^{t} \|\dot{x}(\tau)\|d\tau.$$

Hence, for $t_i + \frac{c}{2k_2} > t > \tilde{t}_i$

$$\|x(t)\|_{\mathcal{A}} \geq \|x(\tilde{t}_i)\|_{\mathcal{A}} - \int_{\tilde{t}_i}^{t} \|\dot{x}(\tau)\|d\tau \geq c - \frac{ck_2}{2k_2} = \frac{1}{2}c.$$

From this we obtain with

$$d = \inf\{\beta(x) | x \in \mathcal{X}, \frac{1}{2}c \leq \|x\|_{\mathcal{A}} \leq k_1\},$$

where $d > 0$ since β is positive definite with respect to the set $\mathcal{A}$, that

$$\infty > \lim_{T\to\infty}\int_0^T \beta(x(t))dt \geq \lim_{N\to\infty}\sum_{i=1}^{N}\int_{\tilde{t}_i}^{\tilde{t}_i+\frac{c}{2k_2}} \beta(x(t))dt \geq \lim_{N\to\infty}\frac{Ndc}{2k_2} \to \infty,$$

which is a contradiction with respect to our initial assumption. Thus $\|x(t)\|_{\mathcal{A}} \to 0$ as $t \to \infty$. ∎

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