Conditions for the existence of a flat input

Steffen Waldherr[†*] and Michael Zeitz[†]

We introduce the concept of an affine flat input to a nonlinear system with a given output function. This approach can be seen as dual to the search for a flat output of a control system with given input. Our results provide a necessary and sufficient condition for the existence of a flat input in the SISO case, which also allows to construct the vector field associated to the flat input. In addition, a relation between the flat input vector field and nonlinear observer design is discussed. A population model is used to illustrate the construction of a flat input.

1 Introduction

After its introduction by Fliess et al. [1992, 1995], the concept of differential flatness of nonlinear systems has attracted great attention in control theory [Martin et al., 1997, Rothfuß, 1997, Delaleau and Rudolph, 1998, Fliess et al., 1999, Martin et al., 2001, Hagenmeyer and Delaleau, 2003, Sira-Ramirez and Agrawal, 2004] and has been beneficial for several industrial control applications as summarised in the introduction of [Rudolph, 2005]. In practical applications, many control systems have been shown to be differentially flat. In most cases, it is required to find the so called flat output for a given control system. The flat output can then be used for feedback stabilisation or trajectory planning with feedforward control in a straightforward way. For outputs other than flat outputs, these approaches are usually much more involved and might even be impossible due to nonminimum phase behaviour or numerical instabilities [Graichen et al., 2005].

When designing a control system, engineers typically have some freedom in choosing both how to implement measurements, i.e. how to choose the output function via the sensor placement, as well as where to place the actuator, i.e. how the input will enter the plant. When applying flatness based control methods, it is desirable to let the

*†Institute for Systems Theory and Automatic Control, Universität Stuttgart, Pfaffenwaldring 9, 70550 Stuttgart, Germany, +49-711-68567657, waldherr@ist.uni-stuttgart.de.

†Institute for System Dynamics, Universität Stuttgart, Pfaffenwaldring 9, 70550 Stuttgart, Germany, +49-711-68566313, michael.zeitz@isys.uni-stuttgart.de

measurement be a flat output. Since methods to determine a flat output for a given control system are available, a reasonable design approach is to place sensors such that the flat output of a previously defined control system is directly measured. This approach is possible in a number of applications.

In this work, we present a dual perspective which might complement the sometimes difficult search for a flat output: given a dynamical system with a defined measured output, the task is now to design an input, or an actuator, such that the measured output is a flat output for the resulting input–output system. Such an input will be called a flat input for the given dynamical system with measurements, or observed system.

In what follows, we investigate the problem of finding a flat input for a given nonlinear system. We present necessary and sufficient conditions for the existence of a flat input. We limit this analysis to SISO systems which are affine with respect to the input. Although in particular the MIMO case would also be of practical interest, we neglect it here due to simplicity in presentation of the results.

The paper is structured as follows. In section 2, we recall some definitions and results which we need for our analysis. Our main result is presented in section 3 and discussed in section 3.2, where we also establish a relation to the design of nonlinear observers with linear error dynamics. An example where the flat input is constructed for a population model with three species is presented in section 4. We conclude with section 5.

2 Preliminaries

2.1 The control and observed system

For input–affine SISO systems of the form

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x), \tag{1}$$

we consider two special cases: the control system

$$\dot{x} = f(x) + g(x)u \tag{2}$$

with no output defined and the observed system

$$\dot{x} = f(x), \qquad y = h(x) \tag{3}$$

without an external input. For all systems, $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}$, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are smooth vector fields and $h : \mathbb{R}^n \to \mathbb{R}$ is a smooth function. Moreover, we assume $f(0) = 0$ and $h(0) = 0$.

In the conditions for existence of a flat input and in its construction, we will make use of the observability matrix for the observed system (3), which is well-known in nonlinear control theory [Hermann and Krener, 1977, Bestle and Zeitz, 1983, Isidori, 1995].

Definition 1. *The matrix*

$$H(x) = \begin{pmatrix} dh(x) \\ dL_f h(x) \\ \vdots \\ dL_f^{n-1} h(x) \end{pmatrix} \tag{4}$$

is called the observability matrix *of the systems* (1) *and* (3)*, where* L_f *represents the Lie derivative of a function along the vector field* f *and* $dL_f h(x)$ *the gradient* $\partial L_f h(x)/\partial x$.

The system is said to satisfy the observability rank condition *if the observability matrix has full rank, i.e.*

$$\operatorname{rank} H(x) = n. \tag{5}$$

The observability rank condition is a necessary condition for existence of the transformation to the observer normal form which gives rise to an observer with linear error dynamics (see also the discussion in Section 3.2).

2.2 Relative degree and flatness

Our view on flatness of SISO systems is by the relative degree of the system [Isidori, 1995]. The relative degree of the input–output system (1) is defined as follows.

Definition 2. *The system* (1) *is said to have* relative degree r *locally at* x_0 *if*

$$\begin{aligned} L_g L_f^i h(x) &= 0 \qquad \text{for } x \in \mathcal{N}(x_0) \text{ and } i = 0, 1, \ldots, r-2 \\ L_g L_f^{r-1} h(x_0) &\neq 0, \end{aligned} \tag{6}$$

where $\mathcal{N}(x_0)$ *is a neighbourhood of* x_0.

Using the observability matrix $H(x)$, (6) can also be written in the form $H(x)g(x) = e_n$, with the vector $e_n = (0, \ldots, 0, 1)^{\mathrm{T}}$. It can be shown that, if the system (1) has a well defined relative degree r, then $0 < r \leq n$. The case where $r = n$ is usually highly favourable in linear and nonlinear control design: as shown within the theory of differential flatness, it implies that the system is invertible in the sense that the state and input can be computed directly from the output and a finite number of its time derivatives.

A SISO system is flat in the sense of Fliess et al. [1995], if and only if it is static feedback linearisable, which in turn is equivalent to having relative degree n. Thus a definition of flatness for the observed and control system (2) and (3) is given as follows.

Definition 3. 1. *The control system* (2) *is said to be flat if there exists an output* $y = \lambda(x)$ *such that the resulting SISO system*

$$\dot{x} = f(x) + g(x)u, \qquad y = \lambda(x) \tag{7}$$

has relative degree n*. In that case,* y *is called a* flat output *of* (2) *defined by the output function* $\lambda(x)$.

2. *The observed system* (3) *is said to be flat if there exists an input vector field* $\gamma(x)$ *such that the resulting SISO system*

$$\dot{x} = f(x) + \gamma(x)u, \qquad y = h(x) \tag{8}$$

has relative degree n*. In that case* u *is called a* flat input *for* (3) *with input vector field* $\gamma(x)$.

With definition 3, flatness is closely related to the exact linearisation of nonlinear systems by state feedback, as treated by Jakubczyk and Respondek [1980] (see also Isidori [1995]). In particular, Jakubczyk and Respondek [1980] have shown that a necessary condition for existence of a flat output for system (2) (and thus existence of a transformation to controller normal form) is that the controllability matrix

$$P(x) = \begin{pmatrix} g(x) & \mathrm{ad}_f\, g(x) & \dots & \mathrm{ad}_f^{n-1}\, g(x) \end{pmatrix} \tag{9}$$

has rank n, where $\mathrm{ad}_f\, g(x)$ represents the Lie product of the vector fields f and g. Notice however, that this condition is not sufficient. A sufficient condition is only given by the involutivity of the distribution spanned by the vector field g and its $(n-2)$-times iterated Lie products with f. Moreover, the computation of the flat output function $\lambda(x)$ requires the solution of a system of partial differential equations. Existence of a flat output is a necessary and sufficient condition for the existence of a transformation of (2) to controller normal form. This transformation can in turn be used to linearise the system by a static state feedback.

3 Main Results

3.1 Existence and computation of the flat input vector field

Our results provide necessary and sufficient conditions for existence of a flat input for a given observed system (3). Based on the observability matrix $H(x)$, we can give an algebraic formula for the vector field $\gamma(x)$ associated to the flat input u in (8).

First, we introduce a lemma which will be useful to show the necessity of the condition we give in the main theorem.

Lemma 1. *If the input–output system* (1) *has relative degree n, then it satisfies the observability rank condition* (5).

Proof. As the observability rank condition (5) is independent of the input u, we can choose $u = 0$ for the proof. For system (1) having relative degree $r \leq n$ at some point x_0, it has already been shown (see e.g. [Isidori, 1995, Lemma 4.1.1]) that the row vectors

$$dh(x_0), dL_f h(x_0), \dots, dL_f^{r-1} h(x_0)$$

are linearly independent. Having $r = n$, we conclude that the observability matrix $H(x)$ has full rank in a neighbourhood of x_0. □

Our main results, the condition for existence of a flat input and the formula to compute it, are presented in the following theorem.

Theorem 1. *The system* (3) *is flat, if and only if it satisfies the observability rank condition* (5). *In that case, the vector field associated to the flat input is given uniquely by*

$$\gamma(x) = \alpha(x) H^{-1}(x) e_n, \tag{10}$$

with a real-valued, nonzero function $\alpha \neq 0$ and the vector $e_n = (0, \dots, 0, 1)^{\mathrm{T}}$.

Remark 1. Theorem 1 provides an easily checkable sufficient and necessary condition for the existence of an input vector field which will give a flat input–output system (8). Moreover, it gives an explicit formula for the flat input vector field $\gamma(x)$, which can be computed without solving any differential equations. It is given by the last column of the inverse observability matrix, possibly scaled by some factor α.

The term *uniquely* is to be understood in the sense that any flat input vector field for system (3) is of the form as given in equation (10). Obviously, the function $\alpha \neq 0$ still provides some degree of freedom.

Proof. First, we prove that the observability rank condition implies flatness of the observed system (3): the observability matrix $H(x)$ being regular implies that we can define the vector field $\tau(x) = H^{-1}(x)e_n$. Consider the SISO system (1) with $g(x) = \tau(x)$. Due to the definition of τ, we have for this system $L_\tau L_f^i h(x) = 0$ for $i = 0, 1, \dots, n-2$ and $L_\tau L_f^{n-1} h(x) = 1$. Obviously, the SISO system has relative degree n. Thus, τ is a flat input vector field for the observed system (3).

Next, we show that the existence of a flat input for system (3) implies that the observability rank condition is satisfied. Let $\bar{\gamma}(x)$ be the vector field associated to the flat input. Then the SISO system (1) with $g(x) = \bar{\gamma}(x)$ has relative degree n (Definition 3). By Lemma 1, it satisfies the observability rank condition, and since this must also hold for $u = 0$, the system (3) satisfies the observability rank condition.

It remains to show that, if the system (3) is flat, than any flat input vector field takes the form as given in (10). Let $\gamma(x)$ be the flat input vector field. Then the relative degree of the system (8) is equal to n. By Definition 2 of the relative degree, this implies that the vector field γ satisfies the system of linear equations

$$\begin{aligned} L_\gamma L_f^i h(x) &= 0, \qquad & i &= 0, 1, \dots, n-2 \\ L_\gamma L_f^{n-1} h(x) &= \alpha(x) \neq 0, \end{aligned}$$

which is equivalent to $\gamma(x)$ being given by equation (10). □

It is of interest to apply our result to linear systems to get a structural insight. Consider the linear observed system

$$\dot{x} = Ax, \qquad y = Cx, \tag{11}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{1 \times n}$. The observability matrix H is constant (not depending on x) and given by

$$H = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}.$$

The system (11) is observable if and only if it H has full rank. In that case, the linear coordinate transformation $z = Hx$ transforms the system to observability normal form

$$\dot{z} = \tilde{A}z, \qquad y = \tilde{C}z, \tag{12}$$

where $\tilde{A}$ is in observability normal form and $\tilde{C} = (1, 0, \dots, 0)$ [Kailath, 1980]. We add a control input u to system (12) by choosing a matrix $\tilde{B} \in \mathbb{R}^{n \times 1}$ and extending the system to

$$\dot{z} = \tilde{A}z + \alpha \tilde{B}u, \qquad y = \tilde{C}z, \tag{13}$$

where $\alpha \in \mathbb{R} \setminus \{0\}$ is a degree of freedom. In fact any nonzero function $\alpha(z)$ could be chosen, but here we want to keep the system linear. The extended system has $y = \tilde{C}z$ as a flat output for $\tilde{B} = (0, \dots, 0, 1)^{\mathrm{T}}$. For the choice $\alpha = 1$, the linear system (13) is represented in the coordinates of the controller normal form. When transforming (13) back to original coordinates x, we find the differential equation

$$\dot{x} = Ax + \alpha H^{-1} \tilde{B}u,$$

where $\alpha H^{-1} \tilde{B}$ is the flat input vector field (compare equation (10)).

3.2 Discussion

What we found to be a flat input vector field is known since quite some time in nonlinear control theory. Due to the duality we mentioned and the existence condition in theorem 1, it is not surprising that the flat input vector field was first used in connection with nonlinear observers [Krener and Isidori, 1983, Bestle and Zeitz, 1983, Krener and Respondek, 1985]. A common approach to the design of a nonlinear observer is to construct linear error dynamics, which is easily done if the observed system is transformed to observer normal form.

Assume such a transformation is possible, and let it be given by $z = \Phi(x)$. Then the system equations read

$$\dot{z} = Az + k(Cz), \qquad y = Cz, \tag{14}$$

where (A, C) are in observer normal form and k is a n-valued function of a real variable. The observer is then designed as

$$\dot{\hat{z}} = A\hat{z} + k(y) + L(C\hat{z} - y) \tag{15}$$

Using different setups, both Krener and Isidori [1983] and Krener and Respondek [1985] have shown how to compute the transformation to observer normal form of system (3): they define a vector field $\tau(x)$ as solution of the linear system of equations

$$\begin{aligned} L_\tau L_f^i h(x) &= 0, \qquad i = 0, 1, \dots, n-2 \\ L_\tau L_f^{n-1} h(x) &= 1. \end{aligned} \tag{16}$$

Obviously, $\tau(x)$ is a flat input vector field for system (3), where the degree of freedom α in (10) has been chosen as $\alpha(x) = 1$. Using the vector field $\tau(x)$, a system of partial differential equation is constructed, the solution of which determines the inverse transformation $x = \Phi^{-1}(z)$ from (14) to (3).

Thus there is some duality between the flat output and the flat input of a nonlinear SISO system, but not a perfect one: for the flat output and the transformation to controller normal form, it holds that

1. the rank condition on the controllability matrix (9) is a necessary condition for existence of a flat output,

2. existence of a flat output is necessary and sufficient for the transformation to controller normal form.

For the flat input and the related transformation to observer normal form, we have shown that

1. the rank condition (5) on the observability matrix (4) is a necessary and sufficient condition for existence of a flat input vector field (10),

2. existence of a flat input vector field (10) is necessary for the transformation of system (3) to observer normal form (14), but not sufficient [Krener and Isidori, 1983].

4 Illustrating example

We illustrate our result using a population model of three species, where two species are neutral to each other and the third species is a predator versus the two others. The model is a generalisation of the predator–prey model introduced by Lotka and Volterra, the type of generalisation we use here has already been studied by Krabs [2003]. The nonlinear model is given by the equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} b_1 x_1 - a_1 x_1 x_3 \\ b_2 x_2 - a_1 x_2 x_3 \\ -b_3 x_3 + a_2 (x_1 + x_2) x_3 \end{pmatrix}, \tag{17}$$

where $x_1, x_2, x_3 \in \mathbb{R}$ represent the population densities of the three species and $b_1, b_2, b_3, a_1, a_2 \in \mathbb{R}_+$ are positive parameters.

We assume that only the total number of prey can be measured, i.e.

$$y = x_1 + x_2. \tag{18}$$

Under mild assumptions on parameter values and the region in state space that we consider, the system (17) with the given output (18) satisfies the observability rank condition (5). Thus we know that a flat input vector field exists. In fact, by a suitable choice of the scaling function $\alpha(x)$, we compute the flat input vector field (10) as

$$\gamma(x) = \begin{pmatrix} a_1 (x_1 + x_2) \\ -a_1 (x_1 + x_2) \\ b_1 - b_2 \end{pmatrix}. \tag{19}$$

If the amount of each species can be manipulated directly, e.g. by adding or removing individual organisms, then $\gamma(x)$ represents an actuator which can be realised (observe also that $\gamma(x)$ depends actually on the output y only). In that case, the resulting SISO system is flat and flatness-based methods can be applied.

5 Conclusions

Our results provide an accessible way to find a flat input for a dynamical system with a predefined single output. Concerning the design of control systems, the search for a flat input is dual to that for a flat output. The determination of a flat input is of practical relevance if the actuators or their position in a plant are not yet fixed which is often the case in biological or network systems. However, some technical constraints on the implementation of an actuator have to be considered when designing the vector field associated to a flat input. Another application with fictitious flat inputs is investigated by Oldenburg and Marquardt [2002] in context with the numerical optimisation of dynamical systems with constraints.

We have also discussed theoretical impacts of our results, in particular a new view on a fictitious vector field which has been used in nonlinear observer design for a long time. Obviously, there is some duality between the flat output with controller normal form and the flat input with observer normal form.

The problem of finding flat inputs for a nonlinear multi output system is more challenging than the single output case. If the system allows for constant Kronecker observability indices, the respective columns of the inverse observability matrix can be used to construct input vector fields which render the input–output system static state feedback linearisable [Krener and Respondek, 1985], which is however not equivalent to flatness in the MIMO case. For general results, also the nonuniqueness of the observability indices will have to be considered. Since no conditions are known for the flatness of general nonlinear MIMO systems, finding flat inputs will be similarly challenging as the construction of flat outputs.

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