Flat inputs in the MIMO case

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Abstract: We extend the notion of flat inputs, which we previously introduced in the SISO case, towards non-linear MIMO systems. For MIMO systems, we have to distinguish two cases for differential flatness, corresponding to feedback-linearizability either by a static or a quasi-static feedback. For the first case, the construction of flat inputs can be solved easily by means of the observability codistribution and indices. The second case remains an open problem, and we illustrate with an example that flat inputs can even be constructed for non-observable systems. In addition, we also discuss the problem of realizing flat inputs as physical actuators in mechanical systems. *Copyright © 2010 IFAC.*

1. INTRODUCTION

The concept of differential flatness of control systems has been introduced by Fliess et al. (1992, 1995) and found great attention in control theory (Martin et al., 1997; Rothfuß, 1997; Delaleau and Rudolph, 1998; Fliess et al., 1999; Martin et al., 2001; Hagenmeyer and Delaleau, 2003; Sira-Ramirez and Agrawal, 2004; Lévine, 2009). Several industrial control applications profited from the application of flatness-based controller design (Rudolph, 2005). Design methods based on the flatness property usually require to determine a so-called flat output: a function of the system's state and possibly inputs such that the state variables and the input trajectories can be parametrised in terms of the flat output trajectory and its derivatives. This property makes the flat output a convenient variable for trajectory planning with feedforward control (Rothfuß et al., 1996). For SISO systems, a flat output is any output yielding a relative degree equal to the system's order, thus making the system amenable to exact feedback linearization (Jakubczyk and Respondek, 1980; Isidori, 1995).

As a dual perspective for the flat output, we have recently introduced the concept of flat inputs (Waldherr and Zeitz, 2008). The construction of a flat output can be understood as a sensor placement problem in order to achieve differential flatness of the resulting input–output system. Dual to this, we have formulated the construction of a flat input as an actuator placement problem in order to achieve the same property. The motivation to construct a flat input for a given output is that with such an input, the tracking problem for the given output can be solved without the consideration of any internal dynamics (Graichen et al., 2005). In the SISO case, it turned out that a flat input can be constructed if and only if the system under consideration satisfies an observability condition. The vector field associated to the flat input, representing an actuator to be implemented, can be computed from a system of linear algebraic equations, and is unique up to a scaling function.

In this paper, we extend the concept of flat inputs towards non-linear MIMO systems. For systems which admit the transformation to an observable form (Krener and Respondek, 1985), flat inputs can be determined similarly to the SISO case. However, in the MIMO case, observability is not necessary for the existence of flat inputs. We provide an example system showing that flat inputs may even exist for non-observable systems in the MIMO case, and that this is related to exact linearization by a quasi-static feedback transformation.

In contrast to the fictitious flat output variables, a flat input must be realized as a physical actuator such that the considered system becomes differentially flat (Waldherr and Zeitz, 2008; Zeitz, 2010). In this paper, we discuss the physical realizability of flat inputs in the case of mechanical systems. Mechanical systems are problematic with respect to realizability of generic actuators constructed in state space, since they are derived from second order constitutive equations and thus obey a specific structure in state space form. To deal with this problem, we propose an algebraic test to check physical realizability of the actuator for the flat input.

The paper is structured as follows. In Section 2, the flatness properties of non-linear MIMO systems are briefly summarized. The main result for flat inputs in the MIMO case is presented in Section 3 and is illustrated by examples. Finally, the realizibility problem of flat inputs as actuators in mechanical systems is discussed in Section 4. We conclude with Section 5.

2. FLATNESS OF MIMO SYSTEMS

Consider the non-linear control system

$$\dot{x} = F(x, u_1, \dots, u_p), \tag{1}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$, and rank $\frac{\partial F}{\partial u} = p$. Whereever derivatives of the inputs occur, we assume that the inputs are sufficiently smooth. The system (1) is said to be differentially flat, if there exist p outputs $z = (z_1, \ldots, z_p)^{\mathrm{T}}$ which satisfy the following conditions (Fliess et al., 1992, 1995).

- The outputs z_i, $i = 1, \ldots, p$, are determined by the state x, the input u, and a finite number of input derivatives:
$$z = \lambda(x, u, \dot{u}, \ldots). \tag{2}$$
- The state x and input u can be parametrized (at least locally) by the outputs z_i, $i = 1, \ldots, p$, and a finite number of output derivatives $\overset{(k)}{z_i}$, $k = 1, \ldots, b_i$, with $\sum_{i=1}^{p} b_i \geq n$:
$$x = \Psi_x(z_1, \ldots, \overset{(b_1-1)}{z_1}, \ldots, z_p, \ldots, \overset{(b_p-1)}{z_p}) \tag{3}$$
$$u = \Psi_u(z_1, \ldots, \overset{(b_1)}{z_1}, \ldots, z_p, \ldots, \overset{(b_p)}{z_p}). \tag{4}$$
These equations define the inverse system for (1), i.e. the state and input are given in terms of the output trajectories, assuming that the output signals are sufficiently smooth.
- The outputs z_i, $i = 1, \ldots, p$ are differentially independent, i.e. they do not satisfy a differential equation of the form
$$\varphi(z, \ldots, \overset{(c)}{z}) = 0. \tag{5}$$

The condition (5) is usually hard to check, but is always satisfied if $\dim z = p$ and conditions (3) and (4) hold.

In the above flatness conditions, we can distinguish two cases concerning the order of the output derivatives in the parametrizations (3) and (4). The first case is under the condition that
$$\sum_{i=1}^{p} b_i = n, \tag{6}$$
while in the second case
$$\sum_{i=1}^{p} b_i > n. \tag{7}$$

2.1 Case I: $\sum_{i=1}^{p} b_i = n$

In the first case (6), the system (1) can be brought into a standard linear form via a static state-dependent input transformation. To this end, we introduce the new inputs w_i, $i = 1, \ldots, p$. Then, the input
$$u = \Psi_u(z_1, \ldots, \overset{(b_1-1)}{z_1}, w_1, \ldots, z_p, \ldots, \overset{(b_p-1)}{z_p}, w_p) \tag{8}$$
transforms the original system (1) to the MIMO Brunovsky normal form
$$\overset{(b_i)}{z_i} = w_i, \qquad i = 1, \ldots, p, \tag{9}$$
yielding p decoupled integrator chains of lengths b_i. In the Brunovsky form, control problems such as trajectory tracking can be solved easily.

Of particular interest in our study are input-affine systems with an output, described by the non-linear differential equation
$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{p} g_i(x) u_i \\ y_i &= h_i(x), \qquad i = 1, \ldots, p, \end{aligned} \tag{10}$$
with $g_i(x)$ the input vector fields and outputs $y_i \in \mathbb{R}$, $i = 1, \ldots, p$. For such systems, the notion of the vector relative degree will be useful in this study.

Definition 1. (Isidori (1995)). The system (10) has a vector relative degree locally at $x_0 \in \mathbb{R}^n$, defined as the p–tuple $r = (r_1, \ldots, r_p)$, if $L_{g_j} L_f^k h_i(x) = 0$ for $j = 1, \ldots, p$, $i = 1, \ldots, p$, $k = 0, \ldots, r_i - 2$, and all x in a neighbourhood of x_0, and the $(p \times p)$ matrix
$$A(x_0) = \begin{pmatrix} L_{g_1} L_f^{r_1-1} h_1(x_0) & \cdots & L_{g_p} L_f^{r_1-1} h_1(x_0) \\ \cdots & \cdots & \cdots \\ L_{g_1} L_f^{r_p-1} h_p(x_0) & \cdots & L_{g_p} L_f^{r_p-1} h_p(x_0) \end{pmatrix}$$
is non-singular.

If the output functions $h_i(x)$ of the system (10) are such that the system has a vector relative degree with
$$\sum_{i=1}^{p} r_i = n, \tag{11}$$
then the outputs y_i are flat outputs. For a given control system without outputs, there exists a complete answer to the question of when output functions satisfying condition (11) exist, involving involutivity conditions of certain distributions generated from the vector fields f and g_i (Jakubczyk and Respondek, 1980; Isidori, 1995).

In the case (6), flatness is also related to observability. Note that in this case, we do not need the dependence on the input u in the definition (2) of the flat output $z = \lambda(x)$. Considering the derivatives of flat outputs $z_i = \lambda_i(x)$, we find that
$$\begin{aligned} z_i &= \lambda_i(x) \\ \dot{z}_i &= L_f \lambda_i(x) \\ &\vdots \\ \overset{(b_i-1)}{z_i} &= L_f^{(b_i-1)} \lambda_i(x), \end{aligned} \tag{12}$$
for $i = 1, \ldots, p$. For ease of notation, let us rewrite (12) as
$$\overset{[b-1]}{z} = q(x). \tag{13}$$
Thereby, the function q is the observability map of the system (10) with respect to the outputs z_i, $i = 1, \ldots, p$. Since the z_i are flat outputs, we also have the state parametrization (3) as the inverse observability map $\Psi_x = q^{-1}$, and the system (10) is observable through the outputs z_i. In this case, observability is independent of the actual input signal.

2.2 Case II: $\sum_{i=1}^{p} b_i > n$

We next consider the case (7), in which the system (1) can not be transformed to a linear system via a static feedback transformation. However, it is possible to apply a quasi-static feedback transformation (Delaleau and Fliess, 1992; Rothfuß, 1997; Delaleau and Rudolph, 1998). To this end, one defines appropriate integers $d_i \geq 0$, $i = 1, \ldots, p$, such that
$$\sum_{i=1}^{p} (b_i - d_i) = n. \tag{14}$$
As in the first case, we introduce the new inputs w_i, $i = 1, \ldots, p$. Using the input

$$u = \Psi_u(z_1, \ldots, \overset{(b_1-d_1-1)}{z_1}, w_1, \ldots, \overset{(d_1)}{w_1}, \ldots, \\ z_p, \ldots, \overset{(b_p-d_p-1)}{z_p}, w_p, \ldots, \overset{(d_p)}{w_p}), \tag{15}$$

the system (1) is transformed to the Brunovsky normal form

$$\overset{(b_i-d_i)}{z_i} = w_i, \qquad i = 1, \ldots, p, \tag{16}$$

yielding p decoupled integrator chains of length $b_i - d_i$. Thus, the flat outputs are also in this case useful to solve typical control problems like trajectory tracking. However, in contrast to the first case, necessary and sufficient conditions for existence of flat outputs have only been proposed recently, and checking these conditions requires the integration of differential forms (Lévine, 2009).

In the case that $\sum_{i=1}^{p} b_i > n$, we do not get a similar relation between flatness and observability as in the first case. In contrast to the previous case, we may need an input dependence in the definition (2). Let us consider flat outputs $z_i = \lambda_i(x, u, \dot{u}, \ldots)$, $i = 1, \ldots, p$ and the map defined by

$$\begin{aligned} z_i &= \lambda_i \\ \dot{z}_i &= L_F \lambda_i + \frac{\partial \lambda_i}{\partial u} \dot{u} + \cdots \\ &\vdots \\ \overset{(b_i-1)}{z_i} &= L_F^{(b_i-1)} \lambda_i + \frac{\partial L_F^{(b_i-2)} \lambda_i}{\partial u} \dot{u} + \cdots , \end{aligned} \tag{17}$$

or shortly

$$\overset{[b-1]}{z} = q(x, u_1, \ldots, \overset{(d_1-1)}{u_1}, \ldots, u_p, \ldots, \overset{(d_p-1)}{u_p}). \tag{18}$$

In this case, the map q transforms into $\mathbb{R}^N$ with $N = \sum_{i=1}^{p} b_i = n + \sum_{i=1}^{p} d_i$, with integers d_i as defined in (14). In contrast to the first case, one can in general not compute the state parametrization Ψ_x in (3) independently of the input parametrization Ψ_u in (4). Thus, observability of the system (1) through the flat outputs (2) will in general depend on the applied input signal.

Example 1. In order to illustrate the input dependency of observability, the flat non-linear system with $n = 3$ and $p = 2$ is considered:

$$\begin{aligned} \dot{x}_1 &= u_1 & z_1 &= x_1 \\ \dot{x}_2 &= x_3 u_1 & z_2 &= x_2 \\ \dot{x}_3 &= u_2. \end{aligned} \tag{19}$$

For this system, the map q defined in (18) has the form

$$\begin{aligned} z_1 &= x_1, & z_2 &= x_2 \\ \dot{z}_1 &= u_1, & \dot{z}_2 &= x_3 u_1, \end{aligned} \tag{20}$$

with $b_1 = b_2 = 2$ and $d_1 = 1$, $d_2 = 0$. The state parametrization Ψ_x is given by

$$\begin{aligned} x_1 &= z_1 \\ x_2 &= z_2 \\ x_3 &= \frac{\dot{z}_2}{\dot{z}_1} \end{aligned} \tag{21}$$

and requires $u_1 = \dot{z}_1 \neq 0$. The latter condition is also necessary for the observability of (19) by the flat outputs z_1 and z_2. Therefore, the state parametrization Ψ_x in (21) depends partly on the input parametrization Ψ_u given by

$$\begin{aligned} u_1 &= \dot{z}_1 \\ u_2 &= \frac{\ddot{z}_2 \dot{z}_1 - \dot{z}_2 \ddot{z}_1}{\dot{z}_1^2}. \end{aligned} \tag{22}$$

3. FLAT INPUTS FOR MIMO SYSTEMS

As a complementary perspective to the flat outputs discussed in the previous section, we consider the problem of finding flat inputs in the MIMO case. This extends our previous results on flat inputs in the SISO case (Waldherr and Zeitz, 2008) towards MIMO systems. Since this is a problem of actuator design, it is reasonable to restrict the discussion to input affine systems of the form (10).

For the definition of flat inputs, consider the observed system

$$\begin{aligned} \dot{x} &= f(x) \\ y_i &= h_i(x), \quad i = 1, \ldots, p, \end{aligned} \tag{23}$$

with $x \in \mathbb{R}^n$ and each $y_i \in \mathbb{R}$. Flat inputs are defined in terms of actuators or vector fields $\gamma_i(x)$ that complement the system (23) to a differentially flat system. As in the previous section, we have to distinguish the two cases (6) and (7).

3.1 Flat inputs in Case I

In the first step, we will restrict the discussion to the first case (6) introduced in Section 2. In this case, flatness of input affine systems (10) can be treated within the classical framework of geometric nonlinear control. Then, we can use the following definition of flat inputs for the system (23).

Definition 2. If there exist p vector fields $\gamma_j(x)$, $j = 1, \ldots, p$ such that the MIMO system

$$\begin{aligned} \dot{x} &= f(x) + \sum_{j=1}^{p} \gamma_j(x) v_j \\ y_i &= h_i(x), \quad i = 1, \ldots, p \end{aligned} \tag{24}$$

locally has a vector relative degree $r = (r_1, \ldots, r_p)$ satisfying $\sum_{i=1}^{p} r_i = n$, then the signals v_j are called *flat inputs* with input vector fields $\gamma_j(x)$.

The computation of vector fields $\gamma_j(x)$ for a flat input will be based on the notion of the observability codistribution and the observability indices, which are defined in the following.

Definition 3. (Krener and Respondek, 1985). The system (23) is said to have *observability indices* $\kappa = (\kappa_1, \ldots, \kappa_p)$ at $x_0 \in \mathbb{R}^n$, if $\sum_{i=1}^{p} \kappa_i = n$, $\kappa_i \geq 0$, $i = 1, \ldots, p$, and there exists a neighbourhood $\mathcal{X}$ of x_0 such that the *observability codistribution*

$$d\mathcal{O}_\kappa = \operatorname{span} \left\{ dL_f^j h_i, 1 \leq i \leq p, 0 \leq j \leq \kappa_i - 1 \right\}. \tag{25}$$

is of constant dimension equal to n in $\mathcal{X}$.

Note that, according to this definition, the observability indices of a given system are not unique. In fact, there may be different p–tuples κ, even when allowing reordering of the outputs, such that $d\mathcal{O}_\kappa$ is of dimension n, see also Example 2 below.

If the system (23) has observability indices κ, we can define p vector fields $\tau_k(x)$, $k = 1, \ldots, p$ as solutions of the pn equations

$$\begin{aligned} L_{\tau_k} L_f^r h_i(x) &= 0 \qquad \text{for } 0 \leq r \leq \kappa_i - 2 \\ L_{\tau_k} L_f^{\kappa_i - 1} h_i(x) &= \delta_{ik} \end{aligned} \tag{26}$$

for $i = 1, \dots, p$, where δ_{ik} is the Kronecker symbol. Due to the dimension condition on $d\mathcal{O}_\kappa$, the vector fields $\tau_k(x)$ as solutions of (26) are unique and correspond to the κ_1-th, $(\kappa_1+\kappa_2)$-th, ..., n-th columns of the inverse observability matrix $(dq(x))^{-1}$. The result on flat inputs for the observed system (23) is then as follows.

Theorem 4. If the system (23) has observability indices κ, then it has flat inputs v_j, $j = 1, \dots, p$ with associated input vector fields $\gamma_j(x)$ satisfying

$$\gamma_j(x) = \sum_{k=1}^{p} \alpha_{kj}(x)\tau_k(x), \quad j = 1, \dots, p, \tag{27}$$

where $\tau_k(x)$, $k = 1, \dots, p$ are the unique solutions of (26) and the $\alpha_{kj}(x)$ are arbitrary scalar functions of the state x such that the matrix

$$A(x) = \begin{pmatrix} \alpha_{11}(x) & \cdots & \alpha_{1p}(x) \\ \cdots & \cdots & \cdots \\ \alpha_{p1}(x) & \cdots & \alpha_{pp}(x) \end{pmatrix} \tag{28}$$

is non-singular.

Proof. To prove that the system (24) has a vector relative degree satisfying condition (11), we make use of the fact that the vector relative degree is invariant under coordinate transformations (Isidori, 1995). Let us introduce coordinates z_{ij}, $i = 1, \dots, p$, $j = 1, \dots, \kappa_i$, which are computed from x as

$$z_{ij} = L_f^{j-1} h_i(x).$$

We denote this transformation by $z = \Phi(x)$. Since the observability codistribution $d\mathcal{O}_\kappa$ has dimension n, Φ is a local diffeomorphism and can be used to transform coordinates. Using the definitions (26) and (27) of the $\tau_k(x)$ and the $\gamma_j(x)$, respectively, the dynamics of the system (24) in z–coordinates write as

$$\begin{aligned} \dot{z}_{i1} &= z_{i2} \\ &\vdots \\ \dot{z}_{i\kappa_i} &= L_f^{\kappa_i} h_i(x) + \sum_{j=1}^{p}\sum_{k=1}^{p} \alpha_{kj}(x) L_{\tau_{\kappa_i}} L_f^{\kappa_i - 1} h_i(x) v_j \\ &= L_f^{\kappa_i} h_i(\Phi^{-1}(z)) + \sum_{j=1}^{p} \alpha_{ij}(\Phi^{-1}(z)) v_j, \end{aligned} \tag{29}$$

for $i = 1, \dots, p$. Obviously, the system (29) has a vector relative degree $r = \kappa$ which satisfies condition (11). Thus the v_j, $j = 1, \dots, p$, with $\gamma_j(x)$ according to (27), are flat inputs for the system (24).

Note that the state x and the input v of the flat system (24) can be parametrized through the measured output y according to (3) and (4) with

$$\begin{aligned} x &= \Phi^{-1}(y_1, \dots, \overset{(\kappa_1-1)}{y_1}, \dots, y_p, \dots, \overset{(\kappa_p-1)}{y_p}) \\ v &= \Psi_v(y_1, \dots, \overset{(\kappa_1)}{y_1}, \dots, y_p, \dots, \overset{(\kappa_p)}{y_p}). \end{aligned} \tag{30}$$

Example 2. We start with a linear example of order $n = 5$ with $p = 2$ outputs, which also illustrates the non-uniqueness of the observability indices. Let the observed system be given by

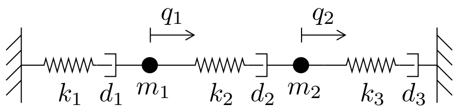

Fig. 1. Mechanical system with two coupled masses

$$\begin{aligned} \dot{x}_1 &= x_2 & y_1 &= x_1 \\ \dot{x}_2 &= x_3 & y_2 &= x_4 \\ \dot{x}_3 &= 0 \\ \dot{x}_4 &= x_5 \\ \dot{x}_5 &= x_2 + x_3. \end{aligned} \tag{31}$$

A straightforward calculation confirms that the tuple $\kappa = (3, 2)$ are observability indices for (31), giving rise to the observability codistribution

$$d\mathcal{O}_\kappa = \text{span}\,\{dx_1, dx_2, dx_3, dx_4, dx_5\}\,,$$

which is of dimension 5. In this case, the solution for (26) is obtained as $\tau_1 = \frac{\partial}{\partial x_3}$ and $\tau_2 = \frac{\partial}{\partial x_5}$. Choosing the matrix $A(x)$ in (28) as identity, we obtain the differentially flat MIMO system

$$\begin{aligned} \dot{x}_1 &= x_2 & y_1 &= x_1 \\ \dot{x}_2 &= x_3 & y_2 &= x_4 \\ \dot{x}_3 &= v_1 \\ \dot{x}_4 &= x_5 \\ \dot{x}_5 &= x_2 + x_3 + v_2 \end{aligned} \tag{32}$$

with flat inputs v_1 and v_2, and vector relative degree $r = (3, 2)$.

However, the tuple $\tilde{\kappa} = (1, 4)$ are also observability indices for (31), giving rise to the observability codistribution

$$d\mathcal{O}_{\tilde{\kappa}} = \text{span}\,\{dx_1, dx_4, dx_5, dx_2 + dx_3, dx_3\}\,.$$

The corresponding solution for (26) is now given by $\tilde{\tau}_1 = \frac{\partial}{\partial x_1}$ and $\tilde{\tau}_2 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}$. Again choosing $A(x)$ as identity, we now obtain the system

$$\begin{aligned} \dot{x}_1 &= x_2 + \tilde{v}_1 & y_1 &= x_1 \\ \dot{x}_2 &= x_3 - \tilde{v}_2 & y_2 &= x_4 \\ \dot{x}_3 &= \tilde{v}_2 \\ \dot{x}_4 &= x_5 \\ \dot{x}_5 &= x_2 + x_3 \end{aligned} \tag{33}$$

with flat inputs $\tilde{v}_1$ and $\tilde{v}_2$, and vector relative degree $r = (1, 4)$. Note that the two cases are structurally different in that they cannot be transformed one into the other by a suitable choice of the matrix $A(x)$.

Example 3. Consider the mechanical system shown in Figure 1. We assume that the masses are $m_1 = m_2 = 1$, non-linear springs generating a force proportional to the cube of the displacement, and linear damping elements. The constitutive equations are given by

$$\begin{aligned} \ddot{q}_1 &= -k_1 q_1^3 - d_1 \dot{q}_1 - k_2(q_1 - q_2)^3 - d_2(\dot{q}_1 - \dot{q}_2) \\ \ddot{q}_2 &= -k_2(q_2 - q_1)^3 - d_2(\dot{q}_2 - \dot{q}_1) - k_3 q_2^3 - d_2 \dot{q}_2. \end{aligned} \tag{34}$$

To derive a state space representation, set $x_1 = q_1$, $x_2 = q_2$, $x_3 = \dot{q}_1$, and $x_4 = \dot{q}_2$. The state space equations are then given by

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -k_1 x_1^3 - d_1 x_3 - k_2(x_1 - x_2)^3 - d_2(x_3 - x_4) \\ \dot{x}_4 &= -k_2(x_2 - x_1)^3 - d_2(x_4 - x_3) - k_3 x_2^3 - d_2 x_4. \end{aligned} \tag{35}$$

Taking the positions as outputs, i.e. $y_1 = x_1$ and $y_2 = x_2$, the system (35) has observability indices $\kappa = (2, 2)$, with the observability codistribution

$$\mathcal{H}_\kappa = \{dx_1, dx_2, dx_3, dx_4\}. \tag{36}$$

The vector fields $\tau_{1,2}$ are computed by use of (26) as $\tau_1(x) = \frac{\partial}{\partial x_3}$ and $\tau_2(x) = \frac{\partial}{\partial x_4}$. For $A(x)$ in (28) as identity, the system (35) has flat inputs given by controlled forces v_1 and v_2 acting on the masses m_1 and m_2, respectively.

3.2 Flat inputs in Case II

We have seen in Section 2 that flatness can either correspond to feedback-linearizability by a static feedback, namely in the case that $\sum_{i=1}^{p} b_i = n$, or to feedback-linearizability by a quasi-static feedback, where $\sum_{i=1}^{p} b_i > n$. However, the systems constructed by addition of flat inputs as in Theorem 4 are always systems where $\sum_{i=1}^{p} b_i = n$. Thus, Theorem 4 still leaves a gap in the construction of flat inputs, since it may be possible to construct flat inputs which fall in the second case even for systems not satisfying the condition in Theorem 4. The following example discusses such a case.

Example 4. Consider the linear observed system with $n = 3$ and $p = 2$ given by

$$\begin{aligned} \dot{x}_1 &= 0 \quad & y_1 &= x_1 \\ \dot{x}_2 &= 0 \quad & y_2 &= x_2 \\ \dot{x}_3 &= 0. \end{aligned} \tag{37}$$

Clearly, system (37) does not have observability indices. However, consider the input vector fields $\gamma_1(x) = \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}$ and $\gamma_2(x) = \frac{\partial}{\partial x_3}$, giving rise to the non-linear MIMO system

$$\begin{aligned} \dot{x}_1 &= v_1 \quad & y_1 &= x_1 \\ \dot{x}_2 &= x_3 v_1 \quad & y_2 &= x_2 \\ \dot{x}_3 &= v_2. \end{aligned} \tag{38}$$

The system (38) is the same as in Example 1 and has already been shown to be flat. To conclude, the observed system (37) has flat inputs v_1, v_2 with the associated vector fields $\gamma_1(x)$ and $\gamma_2(x)$, even though it is not observable. However, in this example, the constructed MIMO system is subject to singularities in the state and input parametrizations.

Example 4 shows that the construction of flat inputs in the MIMO case does not necessarily require observability of the original system. It is thus still an open question what might be necessary conditions for the existence of flat inputs in the MIMO case, and how to determine the associated input vector fields.

4. REALIZABILITY OF FLAT INPUTS IN MECHANICAL SYSTEMS

The physical construction of a flat input in an actual control system requires the implementation of an actuator which corresponds to the vector field associated to the flat input (Waldherr and Zeitz, 2008; Zeitz, 2010). Due to physical and/or technical constraints, this may not be possible for any given system, even if in principle a flat input exists. We refer to this problem as the question of realizability of the flat input.

Realizability of a flat input is of particular relevance for mechanical systems. Mechanical systems use generalized positions, denoted by q, and generalized velocities $\dot{q}$ as state variables. Thereby, the derivative of the position is always equal to the velocity, and it is physically impossible to build an actuator which affects the position directly. As is shown in the following example, this fact may pose a problem for the implementation of a flat input in a mechanical system.

Example 5. Consider the mechanical system described by the linear second-order differential equation

$$\ddot{q} + d\dot{q} + kq = F, \tag{39}$$

where d is a damping constant, k is a spring constant, and F an external controllable force. Assume that the acceleration $y = \ddot{q}$ is measured, thus the output dimension $p = 1$. A state space representation of (39) is obtained by setting $x_1 = q$ and $x_2 = \dot{q}$, yielding

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -kx_1 - dx_2 + F \\ y &= -kx_1 - dx_2. \end{aligned} \tag{40}$$

For $k \neq 0$, the system (40) is observable with observability matrix

$$Q = \begin{pmatrix} -k & -d \\ kd & d^2 - k \end{pmatrix}. \tag{41}$$

Solving the equations (26)–(28) for the flat input vector field yields $\gamma(x) = \alpha(x)(d\frac{\partial}{\partial x_1} - k\frac{\partial}{\partial x_2})$, where $\alpha(x) \neq 0$ is an arbitrary function. For $d \neq 0$, this input is not realizable, since the physical constraint $\dot{x}_1 = x_2$ cannot be affected by an actuator.

Most mechanical systems can be described by a system of second-order differential equations in the generalized positions $q \in \mathbb{R}^m$ as

$$\ddot{q} + D(q, \dot{q}) = F(q, \dot{q})u. \tag{42}$$

Thereby, it is typically possible to control the external forces $F(q, \dot{q})u$, though this may also involve technical constraints which are not addressed in this discussion. Let us assume that we have a measurement given in the form $y = h(q, \dot{q}) \in \mathbb{R}^p$, which includes position, velocity and acceleration measurements. A state space model is constructed by setting $x_p = q$ and $x_v = \dot{q}$, yielding

$$\begin{aligned} \dot{x}_p &= x_v \\ \dot{x}_v &= -D(x_p, x_v) + F(x_p, x_v)u \\ y &= h(x), \end{aligned} \tag{43}$$

with the state variables $x = (x_p^{\mathrm{T}}, x_v^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{2m}$. Note that the input can only enter in the differential equation for x_v. The differential equation for x_p cannot be affected by the input due to the physical constraint that the derivative of the position is equal to the velocity.

We call a given input vector field physically realizable, if it is of the form as in (43), i.e. it acts only on the generalized velocities, not on the generalized positions. In order to characterize the physical realizability of given flat input vector fields, we have the following proposition.

Proposition 5. Define the input distribution

$$\mathcal{I} = \mathrm{span}\{\frac{\partial}{\partial x_{m+1}}, \ldots, \frac{\partial}{\partial x_{2m}}\}. \tag{44}$$

Let $\gamma_j(x)$, $j = 1, \ldots, p$, be flat input vector fields for the system (43). The corresponding flat inputs are physically realizable for the mechanical system (42), if and only if all the vector fields $\gamma_j(x)$, $j = 1, \ldots, p$, satisfy

$$\gamma_j(x) \in \mathcal{I}, \tag{45}$$

with coordinates as in (43).

Proof. If the vector fields $\gamma_i(x)$, $i = 1, \ldots, p$, satisfy the condition (45), then the flat inputs correspond to controlled forces in the constitutive equation (42) and are thus physically realizable. On the other hand, if the condition (45) is not satisfied, then the flat inputs would have to act on the differential equation for the generalized position, and are thus not physically realizable.

Example 6. Consider the mechanical system from Example 3. In this system, any choice of the flat input vector fields $\gamma_1(x) = \alpha_{11}(x)\frac{\partial}{\partial x_3} + \alpha_{12}(x)\frac{\partial}{\partial x_4}$ and $\gamma_2(x) = \alpha_{21}(x)\frac{\partial}{\partial x_3} + \alpha_{22}(x)\frac{\partial}{\partial x_4}$ with a regular matrix $A(x)$ as given in (28) satisfies condition (45) and leads to physically realizable flat inputs in the form of controlled forces.

If the vector fields $\gamma_j(x)$ are constructed by means of $\tau_k(x)$ as defined in (27), the realizability condition (45) can equivalently be applied to the vector fields $\tau_k(x)$, yielding $\tau_k(x) \in \mathcal{I}$, $k = 1, \ldots, p$. The requirement to realize a flat input by a physical actuator means that the duality between a flat output and a flat input is limited to their mathematical determination.

5. SUMMARY AND CONCLUSIONS

We have generalized the notion of flat inputs, introduced in (Waldherr and Zeitz, 2008), towards MIMO systems. It turns out that for systems admitting an observable form, i.e. systems with observability indices according to Definition 3, flat inputs exist and their construction is similar to the SISO case. In particular, one has to solve a system of linear equations for the vector fields $\tau_i(x)$, $i = 1, \ldots, p$, whereby the properties of the observability indices ensure that a unique solution exists. The flat input vector fields $\gamma_i(x)$ are then independent linear combinations of the solution $\tau_i(x)$. From this it follows that the construction of flat inputs for observable systems is significantly easier than finding flat outputs, which requires solving a system of partial differential equations instead of linear equations.

However, the condition that the system has observability indices is not necessary for existence of flat inputs in the MIMO case. To illustrate this, we have presented an example where two input vector fields were found for an unobservable system, such that the controlled system becomes locally observable and differentially flat. Due to the controlled system being only locally observable, the state and input parametrizations possess singularities. As a conclusion, necessary conditions for existence as well as a systematic construction of flat inputs in the general MIMO case are still open problems.

A particular problem of flat inputs, specifically for mechanical systems, concerns the necessity to realize a flat input by physical actuators, which is in contrast to the ficticious nature of flat outputs. In general, the vector fields constructed for the flat input can violate physical constraints, making them infeasible for actual implementation. While it is easily possible to check *a posteriori* whether the vector fields are physically realizable or not, there is not much that one can do if they are not realizable. In this case, also the output would have to be redesigned if one wants to obtain a differentially flat system.

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